Noncommutative tangent bundle: The phase space

by

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Abstract

In this thesis we study the phase space $Ph(A)$ for an associative k-algebra A, with k algebraically closed of characteristic zero (when needed we will assume $k = \mathbb{C}$). The phase space has a universal property analogous to that of the module of Kähler differentials in classical algebraic geometry, and for this and other reasons it can be regarded as a kind of non-commutative (co)tangent bundle. In particular, we include a result showing that with A commutative and smooth, the commutativized version $Ph(A)_{com}$ of $Ph(A)$ will be 'locally trivial'. We also define a cohomology theory for $Ph(A)$ and use it to prove an algebraic variant of an 'inverse function theorem'. Finally we take a short look at representations of the phase space, and how they can be interpreted geometrically.

Contents

Chapter 1

Motivation

1.1 A very short introduction to smooth manifolds

In this chapter we will give some important definitions from differential geometry which in turn will serve as motivation for the phase space construction done in the next chapter. The material presented here can be found (in much more detail!) in e.g. [7] and [8] (which we have used as sources).

Definition 1.1.1. *An n-dimensional (topological) manifold is a topological space M which is second countable and Hausdorff and which is locally Euclidean of dimension n. That is, around each point $p \in M$ we can find a neighbourhood U and a homeomorphism $x : U \to x(U) \subset \mathbb{R}^n$, where $x(U)$ is an open ball in $\mathbb{R}^n$.*

In differential geometry one needs to be able to speak of differentiable/smooth maps between manifolds, and for this to make sense one requires more structure. A topological manifold adorned with such a structure is called a smooth(or C^∞) manifold. Let us proceed with the necessary definitions.

Definition 1.1.2. *Let (x, U), (y, V) be two homeomorphisms as above i.e. U, V are open sets of M with $x(U), y(V)$ open balls in $\mathbb{R}^n$. We say that x and y are C^∞-related if the compositions*

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$
$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

are C^∞ functions.

Definition 1.1.3. *Let M be a manifold. An* ***atlas*** *for M is a collection of mutually C^∞-related homeomorphisms whose domains cover M. Any particular member of the atlas is called a* ***chart****.*

It can be shown that every atlas $\mathcal{A}$ for M is contained in a unique maximal atlas $\mathcal{M}$, so that if $\mathcal{A}'$ is any other atlas containing $\mathcal{A}$, then $\mathcal{A}' = \mathcal{M}$. We have the following definition:

Definition 1.1.4. *A smooth(C^∞) manifold is a topological manifold M together with a maximal atlas for M.*

If $f : M \to N$ is a map between two smooth manifolds, say of dimensions m and n (respectively), what does it mean for f to be smooth/differentiable at a point $p \in M$? The idea is to define it locally: given charts (x, U) and (y, V) for (resp.) p and $f(p)$ we say that f is smooth at p if and only if the composite function $y \circ f \circ x^{-1}$ is C^∞. This makes sense since it is a function between open sets in Euclidean spaces, where the notion of differentiability is well-defined. That this definition is independent of choice of charts follows from the fact that the charts are mutually C^∞-related.

1.2 The tangent bundle in differential geometry

If we have a smooth manifold M and p is any point in M, we can define the tangent space M_p of M at p. Intuitively this is just the vector space consisting of all tangent vectors to the manifold at the given point, but the precise definition is a bit more technical. First let us look at pairs (f, U), where U is an open set containing p and f is a smooth map $f : U \to \mathbb{R}$. We define an equivalence relation on the set of all such pairs by declaring (f, U) and (g, V) to be equal if there is a smaller open set W, $p \in W \subset U \cap V$, such that f and g agree on W. The resulting set of equivalence classes is called the set of germs of C^∞ functions at p, denoted C^∞_p. This can be given a natural structure of an $\mathbb{R}$-algebra.

Definition 1.2.1. *Let C^∞_p be the algebra of germs of C^∞ functions at p. A tangent vector X_p at p is a point-derivation of C^∞_p, that is a linear function $X_p : C^\infty_p \to \mathbb{R}$ satisfying the Leibniz rule: $X_p(fg) = g(p)X_p(f) + f(p)X_p(g)$. The tangent space M_p is then defined as the collection of all tangent vectors at p. It is easily seen to be a (real) vector space.*

Observe that if U is any open set containing p, then U is itself a manifold but due to the local definition of tangent vectors we have $U_p = M_p$. If U is a chart in the atlas for M with corresponding homeomorphism $x : U \to x(U) \subset \mathbb{R}^n$, we are interested in finding an explicit basis for the tangent space.

Definition 1.2.2. *Let $(x, U) = (x^1, x^2, \ldots, x^n)$ be a chart around p, x^i the ith component function of x. Suppose moreover that we have a smooth function f defined on U. The partial derivative of f with respect to x^i is then defined by:*

$$\frac{\partial f}{\partial x^i}(q) := D_i(f \circ x^{-1})(x(q))$$

for $q \in M$. Here D_i is the ordinary ith partial derivative in $\mathbb{R}^n$.

If $\frac{\partial}{\partial x^i}|_p$ is the operator taking a function f to its ith partial derivative, then one can show that this is in fact a tangent vector at p.

Proposition 1.2.3. *Let $p \in M$ and let (x, U) be a chart about p as above. Then a basis for the tangent space M_p is:*

$$\{\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p\}$$

Hence dim M_p=dim M.

Next one can look at the collection of all tangent vectors at all points on the manifold i.e. at the (disjoint) union of all the tangent spaces:

$$TM := \bigsqcup_{p \in M} M_p$$

It is possible to give TM a structure of a smooth manifold. Moreover, if $\pi : TM \to M$ is the natural projection $\pi(M_p) = \{p\}$, then one can show that (TM, M, π) is a so-called C^∞ vector bundle over M. Let us recall the notion of a vector bundle here:

Definition 1.2.4. *A real vector bundle (E, B, π) consists of the following data:*

1. *Topological spaces E (called the total space) and B (the base space)*
2. *A continuous surjection $\pi : E \to B$*
3. *For every $p \in B$ a structure on the fiber $E_p = \pi^{-1}(p)$ of a finite-dimensional real vector space*
4. *A 'local triviality' condition: for every $p \in B$ we can find a neighbourhood U of p, a natural number k, and a homeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that π restricted to each fibre is an isomorphism $\pi^{-1}(q) \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ of vector spaces. The maps ϕ are called **local trivializations** of the bundle.*

A C^∞ vector bundle is just the same concept applied to the category of smooth manifolds (as opposed to the category of topological spaces):

Definition 1.2.5. *Given a smooth manifold M, a C^∞ vector bundle on M is a vector bundle (E, M, π), where E is a smooth manifold the projection map π is smooth and such that the local trivializations are diffeomorphisms.*

One usually requires that the dimension of the fibers should remain constant over the whole base space. If that is the case, we call this common dimension the rank of the vector bundle.

Example 1.2.6. *Let $E := M \times \mathbb{R}^k$, and let $\pi : E \to M$ be projection onto the first factor. Then $(E,\ M,\ \pi)$ is a C^∞ vector bundle of rank k over M. Such a bundle is called a product bundle or a trivial bundle.*

We can also make the collection of all (C^∞) vector bundles into a category and thus specify what it means for two bundles to be equivalent/isomorphic. Let us first define what the morphisms in this category should be:

Definition 1.2.7. *Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles. A **bundle map** (i.e. morphism) from E to F is a pair of smooth maps $(f, \tilde{f})$, $f : M \to N$, $\tilde{f} : E \to F$ such that:*

1. *$\pi_F \circ \tilde{f} = f \circ \pi_E$*
2. *$\tilde{f}$ restricted to each fiber is a linear map of vector spaces*

The first condition means that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & F \\ \downarrow \pi_E & & \downarrow \pi_F \\ M & \xrightarrow{f} & N \end{array}$$

One can then check that this becomes a category, and we make the following definition:

Definition 1.2.8. *A (C^∞) vector bundle over M is called trivial if it is isomorphic to a product bundle over M.*

Hence we see that the last condition in the definition of vector bundles is indeed a local triviality condition; it tells us that locally it is just a product bundle (but not necessarily globally).

Now let us look more closely at the construction of the tangent bundle of a manifold. If $p \in M$ and (x, U) is a chart containing p, we know that any tangent vector X_p at p can be written uniquely as a linear combination

$$X_p = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_p$$

for $a^i \in \mathbb{R}$. Then we can define $\tilde{x} : \pi^{-1}(U) \to x(U) \times \mathbb{R}^n$ by

$$(p, X_p) \mapsto (x^1(p), x^2(p), \ldots, x^n(p), a^1, a^2, \ldots, a^n)$$

This is easily seen to be a bijection, and the idea is then to use $\tilde{x}$ to transfer the topology of $x(U) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ to a topology on $U_p = M_p$ and then use these topologies to get a topology on the disjoint union TM of the tangent spaces. These maps will also serve as the charts in the manifold structure on TM. We will not go further into the details here, but the upshot is that the tangent bundle of M is a C^∞ vector bundle of rank n=dim M over M. Also TM as a smooth manifold has twice the dimension of M.

What is the significance of the tangent bundle? One reason it was introduced was to provide a natural context for speaking of the 'derivative' of a smooth map $f : M \to N$ of manifolds. Suppose we have such a map, and let $p \in M$. Then we can define a linear map $f_{*,p} : M_p \to N_{f(p)}$ between the tangent spaces as follows: If g represents the germ of a smooth function in N then:

$$(f_{*,p}(X_p))g := X_p(g \circ f)$$

Locally, by looking at charts, one can show that $f_{*,p}$ is represented by the Jacobian of the corresponding map between Euclidean spaces. Thus we can indeed view it as a generalization of the total derivative of maps $\mathbb{R}^m \to \mathbb{R}^n$. Also notice that as p runs through all the points of M we end up with a bundle map $(f, f_*) : TM \to TN$.

The tangent bundle also provides the natural framework for defining vector fields on a manifold:

Definition 1.2.9. *A vector field X on a smooth manifold M is a section of the tangent bundle of M. That is, it is a map $X : M \to TM$ such that $\pi \circ X = id$. This just means that to each p in M we get a tangent vector at that given point. If X is smooth, we say the vector field is smooth.*

Now let X be a smooth vector field on M and let $f : M \to \mathbb{R} \in C^\infty(M)$ i.e. a smooth function on M. Then we can define a new function $Xf : M \to \mathbb{R}$ by $(Xf)(p) := X_p(f)$. One can show that we also have $Xf \in C^\infty(M)$. Hence X induces a map $\overline{X} : C^\infty(M) \to C^\infty(M)$.

Lemma 1.2.10. *The map $\overline{X}$ just defined is a derivation of the $\mathbb{R}$-algebra $C^\infty(M)$, i.e. $\overline{X} \in Der(C^\infty(M))$.*

Proof. $\overline{X}$ is obviously linear, since each X_p is linear. Moreover, since each X_p is a point-derivation we see that $\overline{X}(fg)(p) = X_p(fg) = f(p)X_p(g) + g(p)X_p(f) = f(p)\overline{X}(g)(p) + g(p)\overline{X}(f)(p)$. Thus as functions we get $X(fg) = fX(g) + gX(f)$. □

If we let $\mathcal{X}(\mathcal{M})$ denote the set of all smooth vector fields on M we thus have a map $\phi : \mathcal{X}(\mathcal{M}) \to Der(C^\infty(M))$, $\phi(X) = (f \mapsto Xf)$. This is a linear map of real vector spaces, and it can be shown that it is actually an isomorphism. Hence we can identify vector fields on M with derivations of the algebra $C^\infty(M)$.

Next, let us state the inverse function theorem for manifolds.

Theorem 1.2.11. *Let $f : M \to N$ be a smooth map of smooth manifolds. Let $p \in M$, and suppose that the differential $f_{*,p}$ at p is a linear isomorphism of tangent spaces. Then f is a local diffeomorphism at p i.e. there exists an open set U containing p such that $f|_U : U \to f(U)$ is a diffeomorphism.*

1.3 The cotangent bundle and differential forms in differential geometry

Let M be a smooth manifold and let TM be its tangent bundle. One can then define the cotangent bundle, denoted by T^*M, to be the dual bundle of TM. If $\pi : T^*M \to M$ is the projection map then this means that the fibre of a point $p \in M$ is the vector space M_p^* given by

$$M_p^* = Hom(M_p, \mathbb{R})$$

That is, M_p^* is the dual of the tangent space at p, called the cotangent space at p. From this description it follows that we have:

Proposition 1.3.1. *Let $p \in M$ and let (x, U) be a chart about M. Let $dx^i(p) : M_p \to \mathbb{R}$ be defined by $dx_i(\frac{\partial}{\partial x_j}|_p) := \delta^i_j$. Then*

$$\{dx^1(p), \dots, dx^n(p)\}$$

form a basis for the cotangent space M_p^* at p.

Proof. This follows at once from proposition 1.2.3. □

Now let p and (x, U) be as above. If $\omega_p \in M_p^*$ is any cotangent vector at p then by the above proposition it has a unique expression in local coordinates:

$$\omega_p = \sum_{i=1}^{n} a_i dx^i(p)$$

for $a^i \in \mathbb{R}$. Similarly as for the tangent bundle we define $\tilde{x} : \pi^{-1}(U) \to x(U) \times \mathbb{R}^n$ by

$$(p, X_p) \mapsto (x^1(p), x^2(p), \dots, x^n(p), a_1, a_2, \dots, a_n)$$

and it can be proved that this map is a local trivialization making T^*M into a smooth vector bundle over M (by giving T^*M the appropriate topology). Moreover, these maps serve as charts on T^*M, making it into a C^∞ manifold having dimension twice the dimension of M. Now we can define 1-forms on a manifold.

Definition 1.3.2. *A 1-form on a manifold M is a section of the cotangent bundle i.e. a map $\omega : M \to T^*M$ such that $\pi \circ \omega = id_M$. Again, this just means that to each point $p \in M$ we associate a a cotangent vector at that particular point.*

Example 1.3.3. *Let $f : U \to \mathbb{R}$ be a smooth function defined on a chart U around $p \in M$. We define a function $df(p) : M_p \to \mathbb{R}$ by*

$$df(p)(X_p) := X_p(f)$$

This is clearly a linear map, so $df(p)$ is a cotangent vector at p, called the differential of f at p. We wish to find an expression for $df(p)$ in terms of the local coordinates x^i. We know that:

$$df(p) = \sum_{i=1}^{n} a_i dx^i(p)$$

for some constants a_i. Applying $df(p)$ on each tangent (basis) vector $\frac{\partial}{\partial x_j}|_p$ we obtain

$$a_j = df(p)(\frac{\partial}{\partial x_j}|_p) = (\frac{\partial}{\partial x_j}|_p)f = \frac{\partial f}{\partial x_j}(p)$$

Hence

$$df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) dx^i(p)$$

We observe from the above formula that our two possibly different uses of the notation $dx^i(p)$ do, in fact, coincide. Also note that we can define a 1-form df on U by the formula $p \mapsto df(p)$. Below we will see that locally on a manifold, the cotangent space is generated by the differentials dx^i of the (local) coordinate functions.

Definition 1.3.4. *For a manifold M and an open set U we define*

$$\Omega^1(U) := \{\omega | \ \omega \text{ a smooth 1-form on } U\}$$

We define $\Omega^0(U) := C^\infty(U)$. Let $d : \Omega_0(U) \to \Omega_1(U)$ be the map $f \mapsto df$ taking a function f to the 1-form df (defined in the example above).

The map d is called the exterior derivative, and it is merely the first in a sequence of maps, giving rise to a chain complex of $\mathbb{R}$-vector spaces (i.e. $d^k : \Omega^k(U) \to \Omega^{k+1}(U)$ satisfies $d^{k+1} \circ d^k = 0$ for all k):

$$\Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to \ldots \to \Omega^k(U) \to \Omega^{k+1}(U) \ldots$$

Here $\Omega^k(U)$ is the module of (smooth) k-forms on U (which is defined as smooth sections of the kth exterior power of the cotangent bundle of U, not to be defined here). Hence the exterior derivative takes k-forms to $k+1$-forms. One can then study the quotient spaces

$$H^k(U) := kerd^k / imd^{k-1}$$

These spaces measure the extent to which the above sequence fails to be exact, and it is the starting point of the theory of deRham cohomology.

1.4 The tangent space in classical algebraic geometry

In (commutative) algebraic geometry we also have a notion analogous to that of the tangent bundle in differential geometry. The goal of the rest of this chapter is to elaborate on this connection. Some knowledge from algebraic geometry is assumed, for more details see e.g. [3] chapter I.

We look only at the classical case i.e. at the case of varieties over k, where k is an algebraically closed field. If $X \subset \mathbb{A}^n$ is an affine variety and $P \in X$ is any point, then we would like to define the tangent space $T_{X,P}$ of X at P. We have seen that in differential geometry vector fields correspond to derivations of the algebra $C^\infty(M)$, the smooth global functions on the manifold. In the algebraic situation the corresponding object would be the coordinate ring of the variety. Therefore the following definition makes sense:

Definition 1.4.1. *A vector field on X is defined as a k-derivation $D : \mathcal{O}_X \to \mathcal{O}_X$, where $\mathcal{O}_X = A(X)$ is the coordinate ring of X i.e. the ring of global regular functions on X.*

That D is a k-derivation means that it is k-linear, and that it satisfies the Leibniz rule: $D(fg) = fD(g) + gD(f)$.

Example 1.4.2. *What are the possible vector fields on $X = \mathbb{A}^n$? If $D : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ is a vector field, then D is determined by its values $D(x_i)$ on the generators x_i (by applying the Leibniz rule repeatedly). Moreover, if we let $\frac{\partial}{\partial x_i}$ be partial differentiation with respect to x_i, we see that:*

$$D = \sum_{i=1}^{n} D(x_i) \frac{\partial}{\partial x_i}$$

Observe that this is completely analogous to the case of vector fields on a smooth manifold, only that here the coefficients in front of the $\frac{\partial}{\partial x_i}$ *must be polynomials(more generally: rational functions), since we are working in an algebraic setting.*

Suppose now that D is a vector field on X. We would then like to say what it means to evaluate D at a point P of X, and the result should be a tangent vector D_P at that point. The natural thing to do is of course to define $D_P : \mathcal{O}_X \to k$ by $D_P(f) := D(f)(P)$. It is easy to see that $D_P \in Der_k(A(X), k) = Der_k(\mathcal{O}_X, k)$, where the $\mathcal{O}_X$-module structure on $k \cong \mathcal{O}_X/\mathfrak{m}_P$ is given by evalutation at P. Here $\mathfrak{m}_P$ is the maximal ideal at P.

Definition 1.4.3. *The Zariski tangent space* $T_{X,P}$ *of* X *at* P *is defined as the set of all* k*-derivations* $D : \mathcal{O}_X \to k$*. It can easily be shown that this is also a* k*-vector space. Elements of* $T_{X,P}$ *are called tangent vectors at* P*.*

Example 1.4.4. *Again let* $X = \mathbb{A}^n$*. Then our previous example shows that* $T_{X,P}$ *is generated as a* k*-vector space by the tangent vectors* $\frac{\partial}{\partial x_i}|_P$*.*

Since the notion of tangent vectors should be local, it is worth mentioning the following result:

Proposition 1.4.5. $T_{X,P} \cong Der_k(\mathcal{O}_{X,P}, k)$ *as* k*-vector spaces, where* $\mathcal{O}_{X,P}$ *is the local ring at the point* P*.*

Proof. Consider a k-derivation $D : \mathcal{O}_{X,P} \to k$ and let $i : \mathcal{O}_X \hookrightarrow \mathcal{O}_{X,P}$ be the inclusion (since $\mathcal{O}_X$ is an integral domain, X being an affine variety). Then $D \circ i$ is a k-derivation $\mathcal{O}_X \to k$. We claim that the map ϕ taking D to $D \circ i$ is an isomorphism.

Suppose $D \circ i = 0$. Then $D(f) = 0$ for all $f \in \mathcal{O}_X$ (here we identify f with its image $i(f)$ under i). A general element of $\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}$ is of the form $\frac{f}{g}$ for $f, g \in \mathcal{O}_X$, $g(P) \neq 0$. By the quotient rule (which follows from the Leibniz rule):

$$D(\frac{f}{g}) = \frac{gD(f) - fD(g)}{g^2} = \frac{0-0}{g^2} = 0$$

Thus ϕ is injective.

Now suppose we are given a k-derivation $D' : \mathcal{O}_X \to k$. We claim that we can extend it to a k-derivation $D : \mathcal{O}_{X,P} \to k$ i.e. such that $D \circ i = D'$. We define D by:

$$D(\frac{f}{g}) := \frac{gD'(f) - fD'(g)}{g^2}$$

That this is well-defined follows from the proof of proposition 2.1.10. Finally we find that $\phi(D) = D'$ so ϕ is surjective. □

Hence a tangent vector can also be thought of as a derivation of the local ring at P, i.e. the ring of germs of regular functions at P. This is also similar to the case of manifolds, where a tangent vector at a point was defined as a point-derivation of the ring of germs of smooth functions at that point. There is also a more explicit definition of the tangent space.

Proposition 1.4.6. *Let $X \subset \mathbb{A}^n$ be defined by the (polynomial) equations $f_1 = f_2 = \ldots = f_m = 0$. Let $P = (a_1, \ldots, a_n)$ be a point in X. Then the Zariski tangent space at P is given by the points $x = (x_1, \ldots, x_n)$ simultaneously satisfying the following equations:*

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(P)(x_j - a_j) = 0$$

for $i = 1, \ldots, m$. If P is the origin (which we may assume, by a linear change of coordinates), then this is just the kernel of the Jacobian matrix.

Consider the ring $k[\epsilon]/(\epsilon^2)$, where ϵ is a formal variable. This is called the ring of dual numbers, and we have the following result:

Proposition 1.4.7. *For an affine variety X with coordinate ring A, there is a bijection between the set of k-algebra homomorphisms $A \to k[\epsilon]/(\epsilon^2)$ and the set $\{(p, v) \mid v$ is a tangent vector at $p, p \in X\}$.*

Proof. Fix a point $P \in X$, represented by a map $f : A \to k$, and let $D_P \in Der_k(A, k)$ be a tangent vector at P (where as before the module structure on k is given by f). Define $\phi : A \to k[\epsilon]/(\epsilon^2)$ by $\phi(a) := f(a) + D_P(a)\epsilon$. One can check that is is a k-algebra homomorphism.

Conversely, one can also check that every homomorphism $\phi : A \to k[\epsilon]/(\epsilon^2)$ arises in this way. □

There is yet another quite useful way to think of the tangent space and that is in terms of its dual space, which is called the Zariski cotangent space. Let $\mathfrak{m} = \mathfrak{m}_P$ be the maximal ideal corresponding to P. We then have:

Proposition 1.4.8. *$T_{X,P} \cong (\mathfrak{m}/\mathfrak{m}^2)^* = Hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$ as k-vector spaces.*

Proof. We will show that $Der_k(\mathcal{O}_X, k) \cong Hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$ as k-vector spaces. Suppose we are given a derivatation $D : \mathcal{O}_X \to k$. We then define a map $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$ by $\phi(\overline{f}) := D(f)$ for $f \in \mathfrak{m}$. Suppose $f \in \mathfrak{m}^2$. Then f is a finite sum $f = \sum_i g_i h_i$, $g_i, h_i \in \mathfrak{m}$. Hence $\phi(\overline{f}) = D(f) = \sum_i (f_i D(g_i) + g_i D(f_i)) = 0$. Thus ϕ is well-defined, and it is clearly a k-linear map of vector spaces.

Conversely, suppose we are given a linear functional $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$. Now we can identify $\mathcal{O}_X = A(X)$ with $k \oplus \mathfrak{m}$ as a k-vector space. Using this identification, we define $D : \mathcal{O}_X \to k$ by $D(\lambda) = 0$ for $\lambda \in k$ and $D(f) = \phi(\overline{f})$ for $f \in \mathfrak{m}$. One can then check that this is a derivation.

Finally we see that these two maps $Der_k(\mathcal{O}_X, k) \to Hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$ and $Hom_k(\mathfrak{m}/\mathfrak{m}^2, k) \to Der_k(\mathcal{O}_X, k)$ are k-linear maps that are inverses of each other. The result now follows from proposition 1.4.5. □

Let us for a moment go back to the case of a smooth manifold M. Let $C_p^\infty(M)$ be the ring of germs of smooth functions, and consider the map $C_p^\infty(M) \to \mathbb{R}$ given by evaluation at a point $p \in M$. This is a linear map of vector spaces, and the kernel is the maximal ideal (the only one!)

$$\mathfrak{m} := \mathfrak{m}_p := \{f \in C^\infty(M) \mid f(p) = 0\}$$

It can then be shown that the cotangent space M_p^* of M at P is isomorphic to the quotient vector space $\mathfrak{m}/\mathfrak{m}^2$.

Definition 1.4.9. *We say that an affine variety $X \subset \mathbb{A}^n$ is nonsingular at a point $P \in X$ if the dimension of the tangent space $T_{X,P}$ at P equals the dimension of the variety.*

From commutative algebra it is known that the dimension of X can never exceed the dimension of the tangent space, and so the singular points are the points where the tangent space have strictly larger dimension than the variety. Hence from the definition of a regular local ring we see that the following result is true:

Proposition 1.4.10. *An affine variety X is nonsingular at P if and only if the local ring $\mathcal{O}_{X,P}$ is a regular local ring.*

For a general variety (affine, quasi-affine, projective or quasi-projective) we use this criterion as the definition of nonsingularity!

1.5 The (co)tangent bundle in algebraic geometry

Let B be an A-algebra. The module of relative differentials of B over A is defined to be a B-module $\Omega_{B/A}$ together with a derivation $d \in Der_A(B, \Omega_{B/A})$ satisfying the following universal property: for every B-module M and derivations $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $f \circ d = d'$.

There are several ways to construct $\Omega_{B/A}$. One way is simply to consider the free B-module generated by all symbols db for $b \in B$, subject to the relations we want, namely A-linearity and the Leibniz rule:

- $d(ab) = ad(b)$
- $d(bb') = bd(b') + b'd(b)$

One can then verify that this satisfies the universal property. Another way to construct it is given by the next proposition:

Proposition 1.5.1. *Let $\mu : B \otimes_A B \to B$ be the map satisfying $\mu(b \otimes b') = bb'$. Let I be its kernel, and let $d' : B \to I/I^2$ be the map defined by $b \mapsto 1 \otimes b - b \otimes 1$ (mod I^2). Then the pair (I, I^2, d') satisfies the universal property mentioned above and is therefore naturally isomorphic to $(\Omega_{B/A}, d)$.*

Proof. See e.g. [2] p.411. □

We observe that asserting this universal property is the same as asserting that

$$Der_A(B, M) \cong Hom_B(\Omega_{B/A}, M)$$

as functors of M (isomorphism of B-modules). Thus giving an A-derivation from B to M is the same as giving a B-module homomorphism from $\Omega_{B/A}$ to M.

Proposition 1.5.2. *Suppose $B = A[x_1, \ldots, x_n]$. Then $\Omega_{B/A} = \oplus_{i=1}^n B dx_i$.*

Proof. Clearly B is generated as a B-module by the dx_i. We have a map $\psi : B^n \to \Omega_{B/A}$ taking $e_i = (0, \dots, 1, 0, \dots, 0)$ (1 at ith component, 0 otherwise) to dx_i. In order to find an inverse map, consider first the derivations $\frac{\partial}{\partial x_i}$: $B \to B$, $f \mapsto \frac{\partial f}{\partial x_i}$. By the universal property there exists a unique B-module homomorphism $\phi_i : \Omega_{B/A} \to B$ such that $\phi_i \circ d = \frac{\partial}{\partial x_i}$. We can then use these as component functions in order to get a map $\phi : \Omega_{B/A} \to B^n$. It is readily checked that ψ and ϕ are inverses of each other. □

Proposition 1.5.3. *If $A \to B \to C$ are ring homomorphisms, then we have a right-exact sequence of C-modules:*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

Proof. That the last map is surjective is obvious, since it is given by $dc \mapsto dc$.

The first map is given by $1 \otimes db \mapsto d(\psi(b))$, where $\psi : B \to C$ is the given ring homomorphism. Thus we see that the composition $1 \otimes db \mapsto d(\psi(b)) \mapsto d(\psi(b)) \in \Omega_{C/B}$ is zero. Finally, note that $\Omega_{C/A}$ is the same as $\Omega_{C/B}$, only that in the latter we have extra relations $d(\psi(b)) = 0$ for $b \in B$. Hence the kernel of the map $\Omega_{C/A} \to \Omega_{C/B}$ is precisely the image of the first map, and so the sequence is exact. □

Proposition 1.5.4. *Let $\pi : B \to C$ be a surjective homomorphism of A-algebras, $I := ker\pi$. Then there is an exact sequence of C-modules:*

$$I/I^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

where $\delta : I/I^2 \to C \otimes_B \Omega_{B/A}$ is defined by $\delta(\overline{f}) = 1 \otimes df$ for $f \in I \subset B$.

Example 1.5.5. *Let $B = S/I$ for $S = A[x_1, \dots, x_n]$ and an ideal $I = (f_1, \dots, f_m) \subset S$. We wish to calculate $\Omega_{B/A}$. We note that by the second exact sequence given above we have an exact sequence of B-modules:*

$$I/I^2 \to B \otimes_S \Omega_{S/A} \to \Omega_{B/A} \to 0$$

Note that by proposition 1.5.2:

$$B \otimes_S \Omega_{S/A} \cong B \otimes_S S^n \cong B^n$$

is the free B-module generated by the dx_i, $i = 1, \dots, n$. Now the image of the first map $I/I^2 \to B^n$ is generated by the df_i. Hence $\Omega_{B/A} = B^n/(df_i)$.

Since we also have the identities (they follow from the Leibniz rule!)

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

we see that we can identify $\Omega_{B/A}$ with the cokernel of the map $B^m \to B^n$ given by the Jacobian matrix.

Example 1.5.6. *Let $A = k$ and $B = k[x,y]/(y - x^2)$. Then $\Omega_{B/A} = \Omega_{B/k} = Bdx \oplus Bdy/(dy - 2xdx)$*

We have the following result(a proof can be found in Hartshorne [3] p.174):

Proposition 1.5.7. *Let B be a local ring containing a field k isomorphic to its residue field. Assume that k is perfect, and that B is a localization of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to dim B if and only if B is a regular local ring.*

The module of Kähler differential is the algebraic analog of the cotangent bundle in algebraic geometry. This is already suggested by the use of the symbols da (thought of as the differential of a, in analogy with the 1-forms defined on a manifold). Another justification is the following: suppose X is an affine variety with coordinate ring A, and let $\mathfrak{m}$ be a point in X. Then the fiber of Ω at $\mathfrak{m}$ should be defined as the tensor product $\Omega_{A/k} \otimes_A A/\mathfrak{m}$, and one can show that this is isomorphic to the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ at the given point. First note that by proposition 1.5.4 we have an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A A/\mathfrak{m} \to \Omega_{A/\mathfrak{m}/k} \to 0$$

Since $\Omega_{A/\mathfrak{m}/k} = \Omega_{k/k} = 0$, we see that the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A A/\mathfrak{m}$ is surjective, so it remains to show that it is also injective. Dually, we need to show that the map $\delta : Hom_k(\Omega_{A/k} \otimes_A A/\mathfrak{m}, k) \to Hom_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is surjective. Now the term on the left is isomorphic to $Hom_A(\Omega_{A/k}, k) \cong Der_k(A, k)$. One can check that via these identifications, the map δ is given as follows: given a derivation $d : A \to k$ the image $\delta(d)$ is obtained by restricting to $\mathfrak{m}$. In order to show surjectivity, we use the fact that every element in A can be written uniquely in the form $\lambda + c$ for $\lambda \in k$, $c \in \mathfrak{m}$. If we are given a homomorphism $f : \mathfrak{m}/\mathfrak{m}^2 \to k$ we can define $d : A \to k$ by letting $d(\lambda + c) := f(\overline{c})$. One can check that this is a derivation (as in proposition 1.4.8) and that $\delta(d) = f$, hence we are done (this is basically the proof given for proposition 8.7 in [3] p.174).

Of course, there are several noticable differences from the case in differential geometry, the most obvious being that the 'bundle' Ω does not have the same structure as the object it is the 'bundle of' (it is a module, not a ring). Now if we restrict our attention to smooth varieties, it can be shown that the analogy is a lot better; which is quite reasonable since we after all assume our manifolds to be smooth objects. According to proposition 1.5.7 the smoothness assumption entails that our (localized) modules will be free of rank equal to the dimension of the variety, and it can be shown that there is a natural correspondence between such modules (finitely-generated and projective over the coordinate ring of X) and affine vector bundles. In fact, if A is any commutative ring and $X = Spec(A)$ then there is an equivalence of categories between the category finitely-generated projective $A = \Gamma(X, \mathcal{O}_X)$-modules and the category of locally free $\mathcal{O}_X$-modules of finite rank(which again can be shown to correspond to vector bundles), given by $\mathcal{F} \mapsto \Gamma(\mathcal{F}, X)$ for such a locally free sheaf $\mathcal{F}$ (see e.g. the Wikipedia article [9] for more information).

Before proceeding to the next section, let us look at an example showing the failure of the inverse function theorem in algebraic geometry.

Example 1.5.8. *Consider the map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $\phi(x) = x^2$. If $x \neq 0$ then the differential $d\phi$ is an isomorphism. However, ϕ does not have an inverse morphism in a neighbourhood of x, for the function $x \mapsto \sqrt{x}$ is not a morphism in the category of varieties.*

We will return briefly to this topic later, at the end of the next chapter.

Chapter 2

The phase space

2.1 Definitions and basic results

Let A be an associative k-algebra. We let A/k-alg denote the category where the objects are homomorphisms of k-algebras $A \to R$ and where the morphisms are commutative diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ R_1 \arrow[rr] & & R_2 \end{tikzcd}$$

Note that we will often refer to morphisms in this category simply as homomorphisms, and we will often refer to an object $A \to R$ simply as R (for simplicity).

Definition 2.1.1. *Let $i : B \to C$ be a homomorphism of associative k-algebras. We say that a map $D : B \to C$ is a k-derivation if D is k-linear and satisfies the following Leibniz rule:*

$$D(fg) = fD(g) + D(f)g$$

for $f, g \in B$, where on the right side by f and g we mean the images of f and g in C under i.

Motivated by the ideas given in the previous section and following Laudal ([5]), we wish to define an object $A \to Ph(A)$ in this category such that functorially (as sets) we have

$$Der_k(A, -) = Hom_{A/k}(Ph(A), -)$$

That is, we want derivations from A to correspond to homomorphisms from $Ph(A)$. Suppose such an object exists. Then there exists a k-derivation $d : A \to Ph(A)$, corresponding to the identity morphism $Ph(A) \to Ph(A)$, satisfying the following universal propery: For every k-derivation $D : A \to R$ there exists a unique k-algebra homomorphism $\phi : Ph(A) \to R$ such that $\phi \circ d = D$ i.e. such that the following diagram commutes

$$\begin{tikzcd} A \arrow[d, "d"] \arrow[dr, "D"] & \\ Ph(A) \arrow[r, "\phi"] & R \end{tikzcd}$$

The letters Ph are shorthand for 'phase space', and the name stems from Laudal's use of it to model objects in theoretical/mathematical physics.

Next we verify that any such object, if it exists, is characterized up to unique isomorphism by this universal property:

Lemma 2.1.2. *Let $Ph(A)$ be an object satisfying the universal property given above. Then it is unique up to unique isomorphism.*

Proof. Let $\kappa : A \to R$ be another such object, with universal derivation $\delta : A \to R$. Then by the universal property (applied to $Ph(A)$) we have a unique homomorphism $\phi : Ph(A) \to R$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "d"] \arrow[dr, "\delta"] & \\ Ph(A) \arrow[r, "\phi"] & R \end{tikzcd}$$

But we also have a unique homomorphism $\psi : R \to Ph(A)$ such that

$$\begin{tikzcd} A \arrow[d, "d"] \arrow[dr, "\delta"] & \\ Ph(A) & R \arrow[l, "\psi"] \end{tikzcd}$$

commutes. But then the composition $\phi \circ \psi$ is the unique homomorphism $R \to R$ satisfying $(\phi \circ \psi) \circ \delta = \phi \circ (\psi \circ \delta) = \phi \circ d = \delta$, hence it is the identity. Similarly $\psi \circ \phi$ is the identity so we have an isomorphism $Ph(A) \cong R$ (in A/k-alg). □

How can one construct such an object? In general we do it like this: let $Ph(A)$ be the algebra generated by all formal symbols a and da, $a \in A$, subject to the relations we want, namely k-linearity and the Leibniz rule.

We will look more closely at the case where A is finitely generated over k. Then there exists a positive integer n and a surjective homomorphism $\alpha : F \to A$, where $F = k < x_1, x_2, \ldots, x_n >$. Letting I be the kernel we thus have $A \cong F/I$.

We start by letting

$$DF := k < x_1, x_2, \ldots, x_n, dx_1, dx_2, \ldots, dx_n >$$

where the dx_i are formal variables. Clearly we can view F as a subring of DF. Next we define:

$$\overline{DF} := DF/(I, dI)$$

Here we consider I as an ideal of DF in the obvious way i.e. as the 2-sided ideal in DF generated by all elements of $I \subset F \subset DF$. Similarly dI is the 2-sided ideal generated by all expressions df for $f \in I \subset F$, where $d : F \to DF$ is the derivation given by $x_i \mapsto dx_i$.

Observe that the inclusion $F \to DF$ sends the ideal I to zero, as does the derivation $d : F \to DF$. Hence they descend to a homomorphism (an inclusion) $\iota : A \to \overline{DF}$ and a derivation $d : A \to \overline{DF}$.

Theorem 2.1.3. *The object $\iota : A \to \overline{DF}$ together with the derivation d give us $Ph(A)$.*

Proof. To verify this, we need to check that it satisfies the universal property. So let $\kappa : A \to R$ be an object in A/k-alg and let $D : A \to R$ be a derivation. We want to define a homomorphism $\phi : \overline{DF} \to R$ such that $\phi \circ d = D$. Let us first define $\phi' : DF \to R$ by

$$\begin{aligned} x_i &\mapsto \kappa \circ \alpha(x_i) \\ dx_i &\mapsto D \circ \alpha(x_i) \end{aligned}$$

Since ϕ' sends both I and dI to zero, we get a well-defined homomorphism $\phi : \overline{DF} \to R$ satisfying $\phi \circ d = D$. Moreover, we see that if ϕ is going to satisfy this last criterion then we have to define it in this way! Hence ϕ is the unique such homomorphism. □

Example 2.1.4. *Let $A = k < x, y >$ be the free algebra(non-commutative polynomial ring) in two variables. Then $Ph(A) = k < x, y, dx, dy >$ i.e. it is the free algebra in four variables.*

Example 2.1.5. *Let $A = k[x, y]$ be the commutative polynomial ring in two variables. Then $A = k < x, y > /(xy - yx)$ and thus in this case the phase space becomes*

$$\begin{aligned} Ph(A) &= k < x, y, dx, dy > /(xy - yx, dxy + xdy - dyx - ydx) \\ &= k < x, y, dx, dy > /([x, y], [dx, y] + [x, dy]) \end{aligned}$$

Suppose now that $f : A \to B$ is a homomorphism of k-algebras. Since the composition $d \circ f : A \to Ph(B)$ is a k-derivation, we get an induced map $\tilde{f} : Ph(A) \to Ph(B)$. That is, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"] & B \arrow[d, "d"] \\ Ph(A) \arrow[r, "\tilde{f}"] & Ph(B) \end{tikzcd}$$

One can also construct a 'relative phase space', $Ph_A(B)$, such that we have

$$Hom_{B/k}(Ph_A(B), -) = Der_A(B, -)$$

i.e. such that morphisms from $Ph_A(B)$ in the category B/k-alg. correspond bijectively to A-derivations from B. If such an object exist then it will have a universal A-derivation $\delta : B \to Ph_A(B)$ satisfying the following universal property: for every A-derivation $D : B \to C$ there exists a unique morphism $\phi : Ph_A(B) \to C$ such that $\phi \circ \delta = D$. As always, if an object satisfying this property exists, it will be unique up to unique isomorphism. The construction is given as follows:

Proposition 2.1.6. *Let $(df(a))$ be the 2-sided ideal generated by all elements $d(f(a))$ for $a \in A$. If we let $Ph_A(B) := Ph(B)/(df(a))$, then $Ph_A(B)$ will satisfy the universal property described above. Here δ is given by the composition of $d : B \to Ph(B)$ and the projection $\pi : Ph(B) \to Ph_A(B)$.*

Proof. Let $D : B \to C$ be an A-derivation. Then D is also a k-derivation, so by the universal property of $d : B \to Ph(B)$ there exists a unique morphism $\psi : Ph(B) \to C$ satisfying $\psi \circ d = D$. Now $\psi(d(f(a))) = D(f(a)) = 0$, so ψ descends to a unique map $\phi : Ph_A \to C$ from the quotient satisfying $\phi \circ \delta = D$. □

Next we provide a few results concerning the behaviour of the Ph construct with regards to localization. In the next lemma by $Ph(A)_{com}$ we mean the 'commutative' version of $Ph(A)$ i.e. $Ph(A)$ divided out by the commutator ideal.

Lemma 2.1.7. *Let A be commutative. Let $S \subset A$ be multiplicatively closed with $1 \in S$. Then we have:*

$$Ph(S^{-1}A)_{com} \cong S^{-1}Ph(A)_{com}$$

Proof. Define a map $Ph(S^{-1}A) \to S^{-1}Ph(A)_{com}$ by

$$\frac{a}{s} \mapsto \frac{a}{s}$$
$$d(\frac{a}{s}) \mapsto \frac{sda - ads}{s^2}$$

It is obvious that this map respects the relations in $Ph(S^{-1}A)$ i.e. that it respects the Leibniz rule. Moreover, if $\frac{a}{s} = \frac{a'}{s'}$ then $has' = ha's$ for some $h \in S$.

Hence $d(has') = dhas' + h(ads' + s'da) = d(ha's) = dha's + h(a'ds + sda')$.

Thus we get:

$$h(s'da - a'ds) = h(sda' - ads') + dh(a's - as')$$

Multiplying by $ss'h$ yields

$$h^2(s'^2sda - s'a'sds) = h^2(s^2s'da' - sas'ds') + ss'dh(ha's - has')$$

Now using that $has' = ha's$ we get:

$$h^2s'^2(sda - ads) = h^2s^2(s'da' - a'ds')$$

and so

$$h^2(s'^2(sda - ads) - s^2(s'da' - a'ds')) = 0$$

Hence the map is well-defined and we get a homomorphism $\psi : Ph(S^{-1}A)_{com} \to S^{-1}Ph(A)_{com}$. Now define a derivation $\delta : A \to Ph(S^{-1}A)_{com}$ by $\delta(a) = d(\frac{a}{1})$. Then by the universal property of the phase space we have a (unique) homomorphism $Ph(A) \to Ph(S^{-1}A)_{com}$ that takes $a \mapsto \frac{a}{1}$ and $da \mapsto d(\frac{a}{1})$. If we divide out by the commutator ideal we get a well-defined map from $Ph(A)_{com}$, since the ring we are 'going into' is commutative. This map takes elements of S to units, so by the universal property of localization we get a homomorphism $\phi : S^{-1}Ph(A)_{com} \to Ph(S^{-1}A)_{com}$.

Clearly ϕ and ψ are inverses of each other, so we have the desired result. □

In the case of non-commutative rings there are still several ways to define localization, but things are not as simple as in the commutative case. In our case the main problem is that even if A is commutative, a multiplicatively closed subset S in A is not an Ore set in $Ph(A)$ (for a definition of Ore set and a nice historical survey concerning localization, see e.g. [1]). Let us first start with a definition.

Definition 2.1.8. *Let R and B be rings and let $S \subset R$ be multiplicatively closed with $1 \in S$. A homomorphism $f : R \to B$ is called S-inverting if $f(s)$ is a unit in B for every $s \in S$.*

Recall that in the commutative case the localization $S^{-1}A$ of a ring A is characterized by the universal property that any S-inverting homomorphism $A \to B$ factors through the canonical map $A \to S^{-1}A$. A similar result holds also in the general case:

Lemma 2.1.9. *Let R be a ring and $S \subset R$ multiplicatively closed with $1 \in S$. Then there exists a ring R_S and a universal S-inverting homomorphism $R \to R_S$. That is, if we have any S-inverting homomorphism $R \to B$ then there exists a unique map $R_S \to B$ making the diagram commute:*

$$\begin{tikzcd}
R_S \arrow[dr] & \\
R \arrow[u] \arrow[r] & B
\end{tikzcd}$$

Proof. Look at a presentation of R (by generators and relations) and for each $s \in S$ add an element s' and new relations

$$ss' = s's = 1$$

This is the ring R_S. We clearly have a homomorphism $\beta : R \to R_S$, given by $\beta(r) = r$. Now if $f : R \to B$ is S-inverting we define $\phi : R_S \to B$ by $r \mapsto f(r)$ and $s' \mapsto f(s)^{-1}$. This is well-defined because the elements of the form $ss' - 1$ and $s's - 1$ are sent to zero.

Moreover, if the diagram is to commute then we see that is has to be defined this way:

$$\begin{aligned} \phi(ss') = \phi(s)\phi(s') = f(s)\phi(s') = 1 \\ \Rightarrow \phi(s') = f(s)^{-1} \end{aligned}$$

□

Next we give a useful result which will be used in the sequel.

Proposition 2.1.10. *Let A be commutative with $S \subset A$ multiplicatively closed containing 1, and let B be an associative (not necessarily commutative) algebra. Let $f : A \to B$ be an S-inverting ring homomorphism and let $d : A \to B$ be a derivation. Then we can extend d uniquely to a derivation $D : S^{-1}A \to B$.*

Proof. Note first that since f sends every element of S to units in B we have a ring homomorphism $S^{-1}A \to B$. In what follows we shall write the image of an element $\frac{a}{s}$ in B simply as $\frac{a}{s}$, instead of $f(a)f(s)^{-1}$. This works because the image under f in B is a commutative subring of B.

If there is to exist such a derivation D then we need:

$$D(\frac{a}{s}) = aD(\frac{1}{s}) + d(a)\frac{1}{s} = \frac{1}{s}d(a) + D(\frac{1}{s})a$$
$$0 = D(1) = D(\frac{1}{s} \cdot \frac{s}{1}) = \frac{1}{s}d(s) + D(\frac{1}{s})s$$

and therefore

$$D(\frac{1}{s}) = -\frac{1}{s}d(s)\frac{1}{s}$$

Thus we need

$$\begin{aligned} D(\frac{a}{s}) &= -\frac{a}{s}d(s)\frac{1}{s} + d(a)\frac{1}{s} \\ &= \frac{1}{s}d(a) - \frac{1}{s}d(s)\frac{a}{s} \end{aligned} \tag{2.1}$$

To prove that this definiton is well-defined, first note that since $as = sa$ in A we get

$$ad(s) + d(a)s = sd(a) + d(s)a$$

Using this we can show that the two lines in equation (2.1) are equal (by substituting for $ad(s)$). Now suppose $\frac{a}{s} = \frac{b}{t}$ so that $hat = hbs$ in A for some $h \in S$. Then:

$$hd(at) + d(h)at = hd(bs) + d(h)bs$$
$$h(ad(t) + d(a)t - bd(s) - d(b)s) = d(h)(bs - at)$$

From this we find:

$$d(b) = ad(t)\frac{1}{s} + d(a)\frac{t}{s} - bd(s)\frac{1}{s} - \frac{1}{h}d(h)(bs - at)\frac{1}{s}$$

Inserting this into the expression for $D(\frac{b}{t})$ we obtain:

$$\begin{aligned} D(\frac{b}{t}) &= -\frac{b}{t}d(t)\frac{1}{t} + d(b)\frac{1}{t} \\ &= -\frac{b}{t}d(t)\frac{1}{t} + [ad(t)\frac{1}{s} + d(a)\frac{t}{s} - bd(s)\frac{1}{s} - \frac{1}{h}d(h)(bs - at)\frac{1}{s}]\frac{1}{t} \\ &= -\frac{a}{s}d(t)\frac{1}{t} + ad(t)\frac{1}{st} + d(a)\frac{1}{s} - \frac{at}{s}d(s)\frac{1}{st} \\ &= -\frac{a}{s}d(t)\frac{1}{t} + ad(t)\frac{1}{st} + d(a)\frac{1}{s} - \frac{a}{s}[sd(t) + d(s)t - d(t)s]\frac{1}{st} \end{aligned}$$

where in the last two lines we have used that $st = ts$ in order to get the relation $td(s) = sd(t) + d(s)t - d(t)s$, as well as the fact that $\frac{a}{s} = \frac{b}{t}$ and thus $b = \frac{at}{s}$. After clearing up all the smoke we are left with:

$$\begin{aligned} D(\frac{b}{t}) &= -\frac{a}{s}d(s)\frac{1}{s} + d(a)\frac{1}{s} \\ &= D(\frac{a}{s}) \end{aligned}$$

which concludes the proof. □

Proposition 2.1.11. $Ph(S^{-1}A) \cong S^{-1}A *_A Ph(A)$

Here $*_A$ stands for the categorical sum (coproduct) in the category of associative k-algebras. We will not go into the construction here, which is fairly messy, but it is something akin to the free product of groups. Of course, for commutative algebras the coproduct is just the tensor product of algebras.

We will prove the proposition via two lemmas:

Lemma 2.1.12. $S^{-1}A *_A Ph(A) \cong Ph(A)_S$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & S^{-1}A \arrow[d, "\alpha_0"] & \\
A \arrow[ur, "j"] \arrow[r] \arrow[dr, "i"'] & S^{-1}A *_A Ph(A) & Ph(A)_S \\
 & Ph(A) \arrow[u, "\alpha_1"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Now the map α_1 sends $S \subset Ph(A)$ to invertible elements in $S^{-1}A *_A Ph(A)$, hence by the universal property of $Ph(A)_S$ there exists a unique $g : Ph(A)_S \to S^{-1}A *_A Ph(A)$ such that $g \circ \beta = \alpha_1$. Also, the composed map $\beta \circ i$ takes $S \subset A$ to units in $Ph(A)_S$ so by the universal property of the localization we have a map $\gamma : S^{-1}A \to Ph(A)_S$ with $\gamma \circ j = \beta \circ i$. But then we have maps from both $S^{-1}A$ and $Ph(A)$ into $Ph(A)_S$, so by applying the universal property of $S^{-1}A *_A Ph(A)$ we get a map $h : S^{-1}A *_A Ph(A) \to Ph(A)_S$ with $h \circ \alpha_0 = \gamma$, $h \circ \alpha_1 = \beta$.

Thus we can extend our commutative diagram to:

$$\begin{tikzcd}
 & S^{-1}A \arrow[d, "\alpha_0"] \arrow[dr, "\gamma"] & \\
A \arrow[ur, "j"] \arrow[r] \arrow[dr, "i"'] & S^{-1}A *_A Ph(A) \arrow[r, shift left, "h"] & Ph(A)_S \arrow[l, shift left, "g"] \\
 & Ph(A) \arrow[u, "\alpha_1"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Observe that $g \circ h : S^{-1}A *_A Ph(A) \to S^{-1}A *_A Ph(A)$ satisfies $(g \circ h) \circ \alpha_1 = g \circ \beta = \alpha_1$ and $(g \circ h) \circ \alpha_0 = g \circ \gamma = \alpha_0$. But the identity is also a homomorphism satisfying these properties, so by uniqueness (following from the universal property) we require $h \circ g = id$. Similarly the map $h \circ g : Ph(A)_S \to Ph(A)_S$ satisfies $(h \circ g) \circ \beta = h \circ \alpha_1 = \beta$ so by uniqueness we must have $h \circ g = id$, and we are done! □

Lemma 2.1.13. $Ph(S^{-1}A) \cong Ph(A)_S$

Proof. Let $i' : A \to Ph(A)$, $d' : A \to Ph(A)$, $i : S^{-1}A \to Ph(S^{-1}A)$, $d : S^{-1}A \to Ph(S^{-1}A)$ be the homomorphisms and derivations given in the definition of the Ph construction. Again let $\beta : Ph(A) \to Ph(A)_S$ and $j : A \to S^{-1}A$ be the canonical maps. Then the composition $\beta \circ d'$ is a derivation $A \to Ph(A)_S$ and by proposition 2.1.10 it has a unique extension D to a derivation $S^{-1}A \to Ph(A)_S$. Hence by the universal property of $Ph(S^{-1}A)$ there is a unique homomorphism $h : Ph(S^{-1}A) \to Ph(A)_S$ making the following diagram commute:

$$\begin{tikzcd}
S^{-1}A \arrow[d, "d"] \arrow[dr, "D"] & \\
Ph(S^{-1}A) \arrow[r, "h"] & Ph(A)_S
\end{tikzcd}$$

Also, the composition $d \circ j : A \to Ph(S^{-1}A)$ is a derivation which induces a homomorphism $\phi : Ph(A) \to Ph(S^{-1}A)$ with $\phi \circ d' = d \circ j$. Now ϕ takes $S \subset Ph(A)$ to invertible elements, so we have a unique homomorphism $g : Ph(A)_S \to Ph(S^{-1}A)$ such that

$$\begin{tikzcd}
Ph(A)_S \arrow[dr, "g"] & \\
Ph(A) \arrow[u, "\beta"] \arrow[r, "\phi"] & Ph(S^{-1}A)
\end{tikzcd}$$

commutes. Consider now $g \circ h$. If we can show that $g \circ h \circ d = d$ we require

$g \circ h = id$. Similarly, if we can show that $h \circ g \circ \beta = \beta$ we will have to conclude that $h \circ g = id$ and thus these two facts together concludes the proof!

For the first one, note that $g \circ h \circ d \circ j = g \circ D \circ j = g \circ \beta \circ d' = \phi \circ d' = d \circ j$, and this imply our desired result.

For the second: by tracing through the various relationships we see that:

$$h \circ g \circ \beta \circ d' = h \circ \phi \circ d' = h \circ d \circ j = D \circ j = \beta \circ d'$$
$$h \circ g \circ \beta \circ i' = h \circ \phi \circ i' = h \circ i \circ j = \beta \circ i'$$

and since $Ph(A)$ is generated (as a k-algebra) by the images $\mathrm{im}(d')$ and $\mathrm{im}(i')$ we are done. □

Before stating and proving the next proposition(the 'local triviality' result, mentioned in the abstract), we need a lemma from linear algebra. I would like to thank Marc van Leeuwen for providing me with the appropriate hints that helped me formalize everything in the proof (when I asked online).

Lemma 2.1.14. *Let k be a field, and let E be a vector space over k. Suppose we are given m linear functionals $\phi_1, \ldots, \phi_m$ on E. Let $V = \bigcap_{i=1}^m ker(\phi_i)$, and let $\phi \in Hom_k(E, k)$ be another linear functional such that $\phi(V) = \{0\}$. Then ϕ is generated by the ϕ_i.*

Proof. First consider the case $m = 1$. If $V = E$ then both ϕ_1 and ϕ is the zero map and we are done. Thus we may assume that we can find $v \notin V$ i.e. $v \in E$ satisfying $\phi_1(v) \neq 0$. Choose $a \in k$ satisfying $\phi(v) = a\phi_1(v)$. If ϕ is not generated by ϕ_1 then we can find $w \notin V$ such that $\phi(w) \neq a\phi_1(w)$. Then choose $\gamma \in k, \gamma \neq 0$ such that $\gamma\phi_1(w) = \phi_1(v)$. Let $x = \gamma \cdot w - v$. Then by construction $\phi_1(x) = 0$ and so $x \in V$. But $\phi(x) = \gamma\phi(w) - \phi(v) = \gamma\phi(w) - a\phi_1(v) = \gamma\phi(w) - a\gamma\phi_1(w) = \gamma(\phi(w) - a\phi_1(w)) \neq 0$, a contradiction.

Next we claim that we may take the ϕ_i to be linearly independent. For if,say, ϕ_j could be written as a linear combination of the others then $V = \bigcap_{i=1}^m ker(\phi_i) = \bigcap_{i=1, i\neq j}^m ker(\phi_i)$ and so we could throw out ϕ_j and reduce to the $m-1$ case (by induction).

Let us therefore assume linear independence. We then claim that we can choose elements $v_1, \ldots, v_m \in E$ satisfying $\phi_i(v_i) \neq 0$ and $\phi_j(v_i) = 0$ for $j < i$, $i = 1, \ldots, m$. First note that for each i we can choose v_i with $\phi_i(v_i) \neq 0$, for if not then ϕ_i would be the zero map, contradicting the linear independence of the ϕ_i. Fix an index j. Suppose we could not choose the element v_j as described above. Then $\phi_j(v) \neq 0$ would always imply that $\phi_i(v) \neq 0$ for at least one $i < j$. This is equivalent to the following inclusion:

$$\bigcap_{i=1}^{j-1} ker(\phi_i) \subset ker(\phi_j)$$

Hence ϕ_j is a linear functional which vanish on the intersection of the kernels of all the previous ϕ_i, and hence it is generated by them, a contradiction.

We claim that $E = V \cup Span\{v_1, \ldots, v_m\}$. Suppose $v \notin V$. Then $\phi_i(v) \neq 0$ for at least one i. We assume i is the least such index, that is we assume $\phi_k(v) = 0$ for $k < i$. Choose an element $a \in k$ satisfying $\phi_i(v) = a\phi_i(v_i)$. Then by construction $v - av_i$ is in the kernel of ϕ_i. But if $k < i$ then we also have $\phi_k(v - av_i) = \phi_k(v) - a\phi_k(v_i) = 0 - 0 = 0$. Hence $v - av_i \in \bigcap_{j=1}^i ker(\phi_j)$. If it is also in the kernel of the remaining ϕ_j's then we are done. If not, let l be the least index such that $\phi_l(v - av_i) \neq 0$. Applying the same procedure once more, we can choose $b \in k$ with $\phi_l(v - av_i) = b\phi_l(v_l)$. Then the element $v - av_i - bv_l$ will be in the intersection of the kernels $ker(\phi_j)$, $j = 1, \ldots, l$. We continue like this, and sooner or later the process must terminate, since m is finite.

For the final part of the proof, we will construct a linear combination of the ϕ_i's which takes the same values as ϕ on all v_i. We will then have to conclude that this linear combination is equal to ϕ. First choose $a_m \in k$ such that $a_m\phi_m(v_m) = \phi(v_m)$. Next choose a_{m-1} such that $a_{m-1}\phi_{m-1}(v_{m-1}) = \phi(v_{m-1}) - a_m\phi_m(v_{m-1})$. In general, choose a_k such that:

$$a_k\phi_k(v_k) = \phi(v_k) - a_{k+1}\phi_{k+1}(v_k) - a_{k+2}\phi_{k+2}(v_k) - \ldots - a_m\phi_m(v_k)$$

It can then be checked that this choice of a_i's will do the job! □

Suppose A is the coordinate ring of an (irreducible) variety X, which is smooth of dimension d. We then claim we have the following 'local triviality' result:

Proposition 2.1.15. *$Ph(S^{-1}A)_{com} \cong S^{-1}A \otimes_k k[z_1, \ldots, z_d]$ for a localization $S^{-1}A$ of A.*

Proof. Let $A = k[x_1, \ldots, x_n]/I$, where $I = (f_1, \ldots, f_m)$. Since X is smooth of dimension d, the Jacobian matrix has rank $r = n - d$ at every point of X (see [3] ch.I section 5). Now consider the Jacobian matrix $\mathcal{J} = \mathcal{J}(I)$ as a matrix with entries in the quotient field $K = A_{(0)}$ of A. Assume for now that the rank of this matrix is the same as the rank in each point, so that the nullspace has dimension $n - (n - d) = d$ (this will be proved later). Choose a basis for the nullspace, and let $\mathcal{L}$ be the $n \times d$ matrix with these basis vectors as columns. Since the columns are linearly independent, we may without loss of generality assume that $\mathcal{L}$ is of the following form:

$$\mathcal{L} = \begin{pmatrix} I \\ * \end{pmatrix}$$

where I is the $d \times d$ identity matrix. Note that the elements of this matrix are in K, hence they are in a localization $S^{-1}A$ of A which depend on the denominators in this matrix. Now define $\psi : Ph(S^{-1}A)_{com} \to S^{-1}A \otimes_k k[z_1, \ldots, z_d]$ by

$$a \mapsto a \otimes 1$$
$$dx_i \mapsto \sum_{k=1}^{d} \mathcal{L}_{ik} \otimes z_k$$

For elements of $S^{-1}A$ this is clearly well-defined. For a generator f of I we have, since $\mathcal{J}\mathcal{L} = 0$, that:

$$\begin{aligned}\psi(df) = \psi(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i) &= \sum_{i=1}^{n} (\frac{\partial f}{\partial x_i} \otimes 1) \cdot (\sum_{k=1}^{d} \mathcal{L}_{ik} \otimes z_k) \\ &= \sum_{k=1}^{d} (\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathcal{L}_{ik}) \otimes z_k \\ &= \sum_{k=1}^{d} 0 \otimes z_k = 0\end{aligned}$$

Hence ψ is a well-defined homomorphism. By construction it is obvious that ψ is surjective:

$$dx_i \mapsto 1 \otimes z_i \text{ for } i = 1, \ldots, d$$

It remains to show that ψ is injective. First observe that our two rings are graded, the ring on the right side by the z_k's and the ring on the left by the dx_i's (i.e. we let the degree of dx_i be 1 and the degree of everything in $S^{-1}A$ be 0, see next section for more on this). In degree zero ψ is clearly injective. Let us consider the case of degree one next. Let ω be an element of degree 1 i.e. it is of the following form:

$$\omega = f_1 dx_1 + \ldots + f_n dx_n$$

for $f_i \in S^{-1}A$. If $\psi(\omega) = 0$ then:

$$\psi(\omega) = (\sum_{k=1}^{n} f_k \mathcal{L}_{k1}) \otimes z_1 + \ldots + (\sum_{k=1}^{n} f_k \mathcal{L}_{kd}) \otimes z_d = 0$$

Thus

$$\sum_{k=1}^{n} f_k \mathcal{L}_{kj} = 0 \text{ for all } j = 1, \ldots, d \tag{2.2}$$

Now let $\phi_1, \ldots, \phi_m$ be the linear functionals $K^n \to K$ given by the rows of the Jacobian matrix, and let $\phi : K^n \to K$ be the map taking the ith basis vector e_i to $f_i \in K = A_{(0)}$. Let $V = \bigcap_{i=1}^{m} ker(\phi_i) = Nul(\mathcal{J})$ i.e. V is generated by the columns of $\mathcal{L}$. Then the expression 2.2 says that ϕ applied to any vector in V is zero, hence by lemma 2.1.14 ϕ is generated by the ϕ_i:

$$\phi = \sum_{j=1}^{m} a_j \phi_j \text{ for some } a_j \in K$$

Then for all $i = 1, \ldots, n$ we have

$$\begin{aligned} f_i &= \phi(e_i) \\ &= \sum_{j=1}^{m} a_j \phi_j(e_i) \\ &= \sum_{j=1}^{m} a_j \frac{\partial f_j}{\partial x_i} \end{aligned}$$

From this it follows that $\omega = a_1 df_1 + \ldots + a_m df_m$ is in dI i.e. that $\omega = 0$ in $Ph(S^{-1}A)_{com}$. Since all the relations in the ring on the left side lie in degree 1, ψ must be an isomorphism in all other degrees as well. Reason: for any degree > 1 we can define an inverse map (in that degree) taking a generator z_i to $\psi^{-1}(z_i)$.

□

In order to finish the proof completely, we have to prove the assumption we used at the beginning of the proof. We state it as a lemma.

Lemma 2.1.16. *Let X be a variety (irreducible) with coordinate ring A. Let $\mathcal{J}$ be a matrix with entries in A, and suppose that it has rank r at every point $p \in X$. Then it also has rank r considered as a matrix over the quotient field $K = A_{(0)}$ of A.*

Proof. First suppose that the rank is less than r. Then all $r \times r$ minors must vanish in K, hence also in A, that is every $r \times r$ minor m of $\mathcal{J}$ is the zero function. But then $m(p) = 0$ for all $p \in X$ and so the rank is less than r at every point as well, a contradiction!

Next suppose the rank is greater than r. Then we can find an $(r+1)\times(r+1)$ submatrix M of $\mathcal{J}$ having nonzero determinant. Let $F = \det(M)$. Since the rank is r at every point we must clearly have $F(p) = 0$ for all $p \in X$, hence by Hilbert's Nullstellensatz we require $F \in I$ so that $F = 0$ in A, also a contradiction! □

Example 2.1.17. *Let $A = k[x_1, x_2, x_3, x_4]/(x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2)$. This is isomorphic to the subring $k[x^3, x^2y, xy^2, y^3]$ of the integral domain $k[x, y]$, hence it is an integral domain. The Jacobian matrix is as follows:*

$$\begin{pmatrix} x_3 & -2x_2 & x_1 & 0 \\ x_4 & -x_3 & -x_2 & x_1 \\ 0 & x_4 & -2x_3 & x_2 \end{pmatrix}$$

Let us look more closely at the open set U_3 where $x_3 \neq 0$. By Gaussian elimination one can show that the Jacobian in this case can be reduced to

$$\begin{pmatrix} 1 & \frac{-2x_2}{x_3} & \frac{x_1}{x_3} & 0 \\ 0 & 1 & \frac{-2x_2}{x_3} & \frac{x_1}{x_3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

i.e. it has constant rank 2 on U_3 (this is also the largest possible rank). Moreover, as in the proof of the proposition, we find a matrix $\mathcal{L}$ whose columns form a basis for the nullspace. Hence we obtain

$$\mathcal{L} = \begin{pmatrix} \frac{4x_2^2}{x_3^2} & \frac{-2x_1x_2}{x_3^2} \\ \frac{2x_2}{x_3} & \frac{-x_1}{x_3} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and we may define $\psi : Ph(A_{x_3})_{com} \to A_{x_3} \otimes k[z_1, z_2]$ by

$$\begin{aligned} dx_1 &\mapsto \frac{4x_2^2}{x_3^2} \otimes z_1 - \frac{2x_1x_2}{x_3^2} \otimes z_2 \\ dx_2 &\mapsto \frac{2x_2}{x_3} \otimes z_1 - \frac{x_1}{x_3} \otimes z_2 \\ dx_3 &\mapsto 1 \otimes z_1 \\ dx_4 &\mapsto 1 \otimes z_2 \end{aligned}$$

which is an isomorphism.

2.2 Cohomology of Ph(A)

Consider again the the ring $Ph(S) = k < x_1, \dots, x_n, dx_1, \dots, dx_n >$. As in the proof of proposition 2.1.15 we give it a grading by letting

$$\begin{aligned} \deg x_i &:= 0 \\ \deg dx_i &:= 1 \end{aligned}$$

Then $Ph(A) = Ph(S)/(I, dI)$ inherits the grading, since I is generated by homogeneous elements of degree zero and dI is generated by homogeneous elements of degree 1.

We wish to define a linear map $d : Ph(A) \to Ph(A)$ which agrees with the derivation d when restricted to A and which makes $d^2 = 0$. We first define $d : Ph(S) \to Ph(A)$ setting:

$$\begin{aligned} x_i &\mapsto dx_i \\ dx_i &\mapsto 0 \end{aligned}$$

where d satisfies the ordinary Leibniz rule on elements of A, and for $f, g \in A$ let us define

$$\begin{aligned} f dx &\mapsto df dx \\ dx f &\mapsto -dx df \\ f dx g &\mapsto df dx g - f dx dg \end{aligned}$$

Note that this makes sense, since the last definition agrees with the first two when we set (respectively) $g = 1$ and $f = 1$. We extend the definition by recursion. Suppose we have a general homogeneous element of degree > 0 that it is of the form $f dx g$ for some homogeneous $f, g \in Ph(A)$ and some $dx \in \{dx_1, \dots, dx_n\}$. We then set

$$d(f dx g) := df dx g + (-1)^{|f|+1} f dx dg$$

where $|f|$ denotes the degree of f in $Ph(A)$.

Lemma 2.2.1. *With the definition above, d is well-defined.*

Proof. We have defined d unambiguously for elements of degree 0 and 1. Suppose that we have shown d to be well-defined for all degrees less than k, where $k \geq 2$. If x has degree k then it is of the following form:

$$x = f_1 da f_2 db g$$

We have to show that no matter how we place the parantheses we get the same output when applying d, using the above definition. If we write it as $(f_1 da f_2) db g$ we get the following:

$$\begin{aligned} d(x) &= d(f_1 da f_2) db g + (-1)^{|f_1|+|f_2|+2} f_1 da f_2 db dg \\ &= [df_1 da f_2 + (-1)^{|f_1|+1} f_1 da df_2] db g + (-1)^{|f_1|+|f_2|+2} f_1 da f_2 db dg \end{aligned}$$

On the other hand: writing $x = f_1 da(f_2 dbg)$ we obtain:

$$\begin{aligned} d(x) &= df_1 da f_2 dbg + (-1)^{|f_1|+1} f_1 dad(f_2 dbg) \\ &= df_1 da f_2 dbg + (-1)^{|f_1|+1} f_1 da[df_2 dbg + (-1)^{|f_2|+1} f_2 dbdg] \end{aligned}$$

and by multiplying out the parantheses we see that the two expressions are equal! □

Lemma 2.2.2. *$d : A \to Ph(A)$ satisfies $d^2 = 0$.*

Proof. Let us first show this for an element of degree 0. By linearity it suffices to consider elements of the form $f = x_{i_1} \dots x_{i_k}$. For notational conveniency we will (abuse of notation) write $f = x_1 \dots x_k$. The case $k = 1$ follows directly from our definiton of d on generators. Assuming true for $k - 1$ we get:

$$\begin{aligned} d^2(f) &= d(dx_1 x_2 \dots x_k + x_1 d(x_2 \dots x_k)) \\ &= d^2(x_1) x_2 \dots x_k - dx_1 d(x_2 \dots x_k) + dx_1 d(x_2 \dots x_k) + x_1 d^2(x_2 \dots x_k) \\ &= 0 - dx_1 d(x_2 \dots x_k) + dx_1 d(x_2 \dots x_k) + 0 \\ &= 0 \end{aligned}$$

For the general case we can again use an induction argument. Suppose we have proved the result for all degrees less than $k \geq 1$. Let $x = fdag$ be of degree k. Then:

$$\begin{aligned} d^2(x) &= d(dfdag + (-1)^{|f|+1} fdadg) \\ &= d^2(f)dag + (-1)^{|f|+2} dfdadg + (-1)^{|f|+1}[dfdadg + (-1)^{|f|+1} fdad^2(g)] \\ &= 0 + (-1)^{|f|+2} dfdadg + (-1)^{|f|+1} dfdadg + 0 \\ &= 0 \end{aligned}$$

□

Since both the ideal I and the ideal dI is sent to (I, dI), d descends to a map d from the quotient $Ph(A)$ to itself and we get a complex

$$Ph(A)_0 = A \to Ph(A)_1 \to Ph(A)_2 \to Ph(A)_3 \to \dots$$

where $Ph(A)_n$ is the nth graded piece of $Ph(A)$.

Definition 2.2.3. *For d as above we define the ith cohomology group(k-vector space) to be:*

$$H^i(Ph(A), d) := kerd^i / imd^{i-1}$$

for $i \geq 1$. H^0 is defined to be the kernel of the first map $d : A \to Ph(A)$.

Before we study this cohomology, we note the following:

Lemma 2.2.4. *Let $d : Ph(A) \to Ph(A)$ be the operator defined above. If $f : A \to B$ is a homomorphism of algebras, then we have $\tilde{f} \circ d = d \circ \tilde{f}$.*

Proof. Since $\tilde{f}(a) = f(a)$ for all $a \in A$ we see that the result is true for elements of degree zero. Clearly it suffices to prove the lemma for a homogeneous element $\omega \in Ph(A)$ i.e. an element of the form

$$\omega = g_0 dx_{i_1} g_1 dx_{i_2} \dots\ g_{r-1} dx_{i_r} g_r$$

with $g_j \in A$, $dx_{i_j} \in \{dx_1, \dots, dx_n\}$. Then $d\omega = \sum_{k=0}^{r} (-1)^k \omega_k$, where

$$\omega_k = g_0 dx_{i_1} g_1 \dots\ dx_{i_k} dg_k \dots g_{r-1} dx_{i_r} g_r$$

i.e. ω_k is the same as ω, but with a d in front of g_k. Since $\tilde{f}$ and d commute on elements of A we get:

$$\tilde{f}(\omega) = \tilde{f}(g_0) d\tilde{f}(x_{i_1}) \dots\ d\tilde{f}(x_{i_r}) \tilde{f}(g_r)$$

Hence $d(\tilde{f}(\omega)) = \sum_{k=0}^{r} (-1)^k \eta_k$, where

$$\eta_k = \tilde{f}(g_0) d\tilde{f} x_{i_1} \dots\ d\tilde{f} x_{i_k} d\tilde{f}(g_k) \dots \tilde{f}(g_r)$$

Finally $\tilde{f}(d\omega) = \sum_{k=0}^{r} (-1)^k \tilde{f}(\omega_k)$, and we see that $\tilde{f}(\omega_k) = \eta_k$ (again using that $d\tilde{f}(g_k) = \tilde{f}(dg_k)$), thus finishing the proof. □

Now let us return to the study of the cohomology spaces $H^k(Ph(A))$. We claim that it is actually acyclic i.e. that the cohomology vanishes for all $k \geq 1$.

Proposition 2.2.5. *Let $S = k < x_1, \dots, x_n >$ and let $A = S/I$ where $k = \mathbb{C}$. Then $H^i(Ph(A), d) = 0$ for $i \geq 1$ and $H^0(Ph(A), d) = k$.*

In order to prove the proposition, consider first the case of $Ph(S)$. For $r \geq 1$ we define $s : Ph(S)_r \to Ph(S)_{r-1}$ by k-linearity and the following formula on a monomial

$$\omega = f_0 dx_{i_1} f_1 dx_{i_2} f_2 \cdot \ldots \cdot dx_{i_r} f_r$$

of degree r in $Ph(S)$ (where we assume that the f_i are monomials in $x_1, \dots, x_n$):

$$s(\omega) := \sum_{j=1}^{r} (-1)^{j+1} \omega_j$$

where ω_j is the same as ω but with the d in front of dx_{i_j} removed. Observe that if ω_1 and ω_2 are two such monomials then:

$$s(\omega_1 \omega_2) = s(\omega_1)\omega_2 + (-1)^{|\omega_1|} \omega_1 s(\omega_2)$$

If we want to define s on all of $Ph(S)$ we can let it take every polynomial in S to zero.

One can also show that d satisfies the same formula i.e. that

$$d(\omega_1 \omega_2) = d(\omega_1)\omega_2 + (-1)^{|\omega_1|} \omega_1 d(\omega_2)$$

This fact will be used in the next proof.

Lemma 2.2.6. *Consider now the grading on $Ph(S)$ given by deg $dx_i = 1$ and deg $x_i = 1$. If ω is a monomial as above (i.e. a monomial in the x_i and the dx_i) of degree $d^0(\omega)$ with respect to this grading, then $(sd+ds)(\omega) = d^0(\omega)\cdot\omega = \omega+\omega+\ldots+\omega$ ($d^0(\omega)$ times).*

Proof. We use induction on the degree of ω in $Ph(S)$ (the old grading). If the degree is 1 then we can assume ω is of the form $\omega = fdxg$ where f and g are monomials in the x variables. Then one easily checks that $s(df) = d^0(f)\cdot f$, and so we get:

$$\begin{aligned}
(ds+sd)(\omega) &= d(s(\omega)) + s(d(\omega)) \\
&= d(fxg) + s(dfdxg - fdxdg) \\
&= dfxg + fdxg + fxdg + s(df)dxg - dfxg - fxdg + fdxs(dg) \\
&= \omega + d^0(f)\cdot\omega + d^0(g)\cdot\omega \\
&= (d^0(f) + d^0(g) + 1)\cdot\omega \\
&= d^0(\omega)\cdot\omega
\end{aligned}$$

Now suppose ω has degree ≥ 2. Then we can write $\omega = \omega_1\omega_2$ i.e. as a product of two monomials of less degree.

$$\begin{aligned}
(ds+sd)(\omega) &= d(s(\omega)) + s(d(\omega)) \\
&= d(s(\omega_1\omega_2)) + s(d(\omega_1\omega_2)) \\
&= d(s(\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 s(\omega_2)) + s(d(\omega_1)\omega_2 + (-1)^{|\omega_1|}\omega_1 d(\omega_2)) \\
&= (ds)(\omega_1)\omega_2 + (-1)^{|\omega_1|-1}s(\omega_1)d(\omega_2) + (-1)^{|\omega_1|}d(\omega_1)s(\omega_2) \\
&+ (-1)^{2|\omega_1|}\omega_1(ds)(\omega_2) + (sd)(\omega_1)\omega_2 + (-1)^{|\omega_1|+1}d(\omega_1)s(\omega_2) \\
&+ (-1)^{|\omega_1|}s(\omega_1)d(\omega_2) + (-1)^{2|\omega_1|}\omega_1(sd)(\omega_2) \\
&= (ds+sd)(\omega_1)\omega_2 + \omega_1(ds+sd)(\omega_2)
\end{aligned}$$

By induction this reduces to:

$$\begin{aligned}
(ds+sd)(\omega) &= (d^0(\omega_1) + d^0(\omega_2))\cdot\omega \\
&= d^0(\omega)\cdot\omega
\end{aligned}$$

□

Now we can finish the proof of the propositition in the case of $Ph(S)$. If $\omega \in Ph(S)_r$ with $d(\omega) = 0$, write ω as a sum of graded components according to the 'new' grading introduced above: $\omega = \sum_i \omega_i$. Then $d(\omega_i) = 0$ for each i, so by linearity we can reduce to the case where ω is homogeneous in this grading. But then the above lemma shows that:

$$\begin{aligned}
(ds+sd)(\omega) &= d(s(\omega)) + s(d(\omega)) = d(s(\omega)) \\
&= d^0(\omega)\cdot\omega
\end{aligned}$$

Since $k = \mathbb{C}$ we see that $\omega = d(\frac{1}{d^0(\omega)}s(\omega))$ is in the image, so the cohomology vanishes.

Consider the general case where $A = S/I$. If the ideal I is homogeneous, one can show that the homotopy s descends to the quotient $Ph(A) = Ph(S)/(I, dI)$, and thus the same proof can be used in this case. However, it does not work in the completely general situation. In order to finish the proof we will start by proving the following lemma:

Lemma 2.2.7. *Suppose $\omega \in Ph(S)_r$ satisfies $d\omega \in (I, dI)$, $r \geq 1$. Then we can find an element $\omega' \in Ph(S)$ such that:*

1. *ω and ω' are equal in lowest degree*
2. *the degree of ω' does not exceed that of ω*
3. *$\omega' = d\phi$ modulo the ideal (I, dI), for some element $\phi \in Ph(S)$*

Proof. First note that by degree we here (again) mean the standard grading on the polynomial ring $Ph(S)$. Also note that we may clearly assume $d\omega \neq 0$ for if $d\omega = 0$ then $\omega = d\phi$ for some ϕ (by the case of the free algebra S).

Since $d\omega$ is in the ideal (I, dI) we know that it has the following form:

$$d\omega = \sum \xi f \eta + \sum \xi' df' \eta'$$

for various $\xi, \xi', \eta, \eta' \in Ph(S)$, $f, f' \in I \subset S$.

Let $(d\omega)_0$ be the term of lowest degree in $d\omega$. We may assume that $(d\omega)_0 = d(\omega_0)$, where ω_0 is the term of lowest degree in ω. Reason: if not, then $\omega = \rho + \eta$ with $d\rho = 0$, and where the lowest degree η_0 of η satisfies $d\eta \neq 0$ (since $d\omega \neq 0$). We can then replace ω with $\omega - \rho$.

By lemma 2.2.6 we have

$$d^0(\omega_0) \cdot \omega_0 - ds(\omega_0) = sd\omega_0 = s((d\omega)_0)$$

Now what does the lowest degree term of $d\omega$ look like? If we let ξ_0, f_0, η_0 denote the terms of lowest degree in ξ, f, η (respectively), then it will be of the form:

$$(d\omega)_0 = \sum \xi_0 f_0 \eta_0 + \sum \xi'_0 df'_0 \eta'_0$$

where in the first sum we sum over all ξ, f, η satisfying $|\xi_0| + |f_0| + |\eta_0| = |(d\omega)_0|$ and in the second over all ξ', df', η' satisfying $|\xi'_0| + |df'_0| + |\eta'_0| = |(d\omega)_0|$. Since each f'_0 is homogeneous we get:

$$\begin{aligned} s((d\omega)_0) = &\sum (s(\xi_0) f_0 \eta_0 + (-1)^{|\xi_0|} \xi_0 f_0 s(\eta_0)) + \\ &\sum (s(\xi'_0) df'_0 \eta'_0 + (-1)^{|\xi'_0|} d^0(f'_0) \cdot \xi'_0 f'_0 \eta'_0 + (-1)^{|\xi'_0|+1} \xi'_0 df'_0 s(\eta'_0)) \end{aligned}$$

Now let us define ζ by:

$$\begin{aligned} \zeta = &\sum (s(\xi) f \eta + (-1)^{|\xi|} \xi f s(\eta)) + \\ &\sum (s(\xi') df' \eta' + (-1)^{|\xi'|} d^0(f'_0) \cdot \xi' f' \eta' + (-1)^{|\xi'|+1} \xi' df' s(\eta')) \end{aligned}$$

This is essentially just the same expression as above, except we have removed the zeroes in almost all places. We see that ζ in lowest degree equals the term of lowest degree in $s(d\omega)$. Moreover, it is clearly in the ideal (I, dI).

Define ω' by:

$$\omega' := \frac{1}{d^0(\omega_0)} \cdot (d(s\omega) + \zeta)$$

Then by construction ω' and ω are equal in lowest degree, and by letting $\phi := \frac{1}{d^0(\omega_0)} \cdot s\omega$ we see that $\omega' = d\phi$ modulo (I, dI). Clearly the degree of ω' does not exceed that of ω. □

We are now in a position to prove the proposition in the general case.

Proof. (Proposition 2.2.5) We are given $A = S/I$, $Ph(A) = Ph(S)/(I, dI)$. Let $r \geq 1$ and let $\overline{\omega} \in Ph(A)_r$ be such that $d(\overline{\omega}) = \overline{d\omega} = \overline{0}$. Then $d\omega \in (I, dI)$ in $Ph(S)$. We will construct an element $\phi \in Ph(S)$ satisfying $\omega = d\phi$ modulo (I, dI), thus showing that $\overline{\omega} = d(\overline{\phi})$.

Let ω_0 be the term of lowest degree in ω. Let ω_1 be an element having the three properties listed in lemma 2.2.7. Let $\omega^1 := \omega - \omega_1$. We have $d\omega^1 = d\omega$ modulo (I, dI), so $d\omega^1$ is in (I, dI) also. Applying lemma 2.2.7 once more, this time to ω^1, we obtain an element ω_2 equal to ω^1 in lowest degree and such that $\omega_2 = d\phi_2$ modulo (I, dI) for some ϕ_2 etc. Let $\omega^2 := \omega^1 - \omega_2$. We continue this process. Since in each step, we 'remove' the term of lowest degree, the process will eventually terminate, and we will end up with $\omega^N = 0$ for some N. Then:

$$\omega = \omega_1 + \omega_2 + \ldots + \omega_N = d(\phi_1) + \ldots + d(\phi_N) + \text{ something in } (I, dI)$$

Finally let $\phi := \phi_1 + \ldots + \phi_N$. □

2.3 An algebraic inverse function theorem

We can use the vanishing of the cohomology to prove the following algebraic variant of an 'inverse function theorem':

Proposition 2.3.1. *Let $f : A \to B$ be a k-algebra homomorphism such that $\tilde{f} : Ph(A) \to Ph(B)$ is injective. Then the following two statements are equivalent:*

1. *f is surjective*
2. *$\tilde{f}$ is surjective on differentials*

Proof. (1)⇒(2): For b in B we have $b = f(a)$ for some $a \in A$. Hence $db = df(a) = \tilde{f}(da)$.

(2)⇒(1): Let $b \in B$. By assumption $db = \tilde{f}(z)$ for some $z \in Ph(A)$. Since $0 = d^2 b = d(\tilde{f}(z)) = \tilde{f}(dz)$, we get $dz = 0$ by the injectivity of $\tilde{f}$. By proposition 2.2.5 we must have $z = dg$ for some $g \in Ph(A)_0 = A$. Thus $db = \tilde{f}(dg) = d(\tilde{f}(g)) = d(f(g))$ and so $b - f(g) \in H^0(Ph(B)) = k$. Thus $b = f(g) + \lambda = f(g) + f(\lambda) = f(g + \lambda)$ for $\lambda \in k$ and f is surjective. □

Example 2.3.2. *Let $A = k[t]$, $B = k[x, y]/(y - x^2)$ and let $f : A \to B$ be the map $t \mapsto y$. Then $Ph(A) = k < t, dt >$, $Ph(B) = k < x, y, dx, dy > /(y - x^2, dxy + xdy - dyx - ydx, dy - xdx - dxx)$ and $\tilde{f} : Ph(A) \to Ph(B)$ is given by $t \mapsto y$, $dt \mapsto dy$. Now a short computation shows that $\Omega_{B/A} = 0$, yet f is clearly not an isomorphism (it is not surjective).*

In fact, this is the typical example of what is called an etalé map, and the Ω-construction cannot distinguish between isomorphisms and etalé maps. The reason for this is that in classical algebraic geometry we do not have the inverse function theorem (see discussion in chapter 1). However, we see from proposition 2.3.1 that $\tilde{f}$ can detect whether f is surjective or not, and so it makes sense regard that proposition as a kind of 'algebraic inverse function theorem'.

For the record we will provide one definition of etalé maps, in the case of affine varieties:

Definition 2.3.3. *Let $\phi : X \to Y$ be a morphism. If $p \in X$ is a non-singular point, we say that ϕ is etalé at p if the differential $d\phi : T_pX \to T_{\phi(x)}Y$ is an isomorphism of vector spaces.*

Chapter 3

Representation theory

3.1 Basic definitions and results

In this section we will give a short introduction to the main definitions from the representation theory of associative algebras over k. Afterwards we will take a slight 'detour' and introduce the so-called trace ring (see e.g. [4] for more details and examples than provided here) as a tool to study (certain kinds of) representations, before we round up the thesis by looking at some examples of representations of $Ph(A)$.

Definition 3.1.1. *Let A be a k-algebra. A representation of A is a k-vector space V together with a k-algebra homomorphism $\rho : A \to End(V)$.*

Observe that such a map ρ corresponds to a left A-module structure on V, given by $a \cdot v := \rho(a)(v)$. Now if V is finite-dimensional, say of dimension n, then $End(V)$ is isomorphic to $M_n(k)$, the ring of all $n \times n$ matrices with entries in k. Hence we can view a representation of A as assigning to each element in A a matrix.

Example 3.1.2. *Let $V = A$ and let $\rho : A \to End(A)$ be given by multiplication in A i.e. by $\rho(a)(b) := ab$. This is called the regular representation of A.*

Example 3.1.3. *$A = k$. In this case a representation of A is simply a k-vector space.*

Example 3.1.4. *$A = k < x_1, x_2, \dots, x_m >$. Then an n-dimensional representation V of A is uniquely determined by the (arbitrary) choice of m $n \times n$-matrices, that is by assigning to each variable x_i a matrix in $M_n(k) \cong End(V)$.*

Definition 3.1.5. *Let $\rho : A \to End(V)$ be a representation of A. A subrepresentation is given by a subspace $W \subset V$ which is invariant under the action of $\rho(A)$ i.e. which satisfies $\rho(a)(W) \subset W$ for all $a \in A$. We say that ρ is irreducible or simple if it is nonzero and has no proper nontrivial subrepresentations i.e. if $V \neq 0$ with 0 and V as the only subrepresentations.*

Note that a subrepresentation corresponds to giving the subspace W a submodule structure inherited from the A-module structure on V. Also observe that if we have two representations V_1, V_2 of A then the k-vector space $V_1 \oplus V_2$ also becomes a representation in a natural way: the module strucure is given by the standard module structure of direct sum of two modules.

Definition 3.1.6. *A representation $\rho : A \to End(V)$ is indecomposable if it cannot be written as a direct sum of two proper, non-trivial subrepresentations. Otherwise we say it is decomposable.*

We see immediately that an irreducible representation is also indecomposable (but the converse is not true in general).

Definition 3.1.7. *A representation is semi-simple if it can be written as a direct sum of irreducible subrepresentations. In particular, a simple representation is also semi-simple.*

Definition 3.1.8. *Let V_1, V_2 be two representations of A. A homomorphism/interwining operator $\phi : V_1 \to V_2$ is a linear transformation of the underlying vector spaces which also commutes with the action of A i.e. which satisfies $\phi(av) = a\phi(v)$ for all $a \in A$ and $v \in V_1$.*

Formally, if our two representations are given by $\rho : A \to End(V_1)$ and $\rho' : A \to End(V_2)$, then ϕ is a linear transformation such that for all $a \in A$ the following diagram commutes:

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho(a)"] & V_2 \arrow[d, "\rho'(a)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

We say that such an intertwining operator ϕ is an isomorphism if it is an isomorphism of the underlying vector spaces.

We are often interested in classifying representations of A up to isomorphism. Now when does this happen? If $\phi : V_1 \to V_2$ is an isomorphism then (see diagram above) $\rho'(a) \circ \phi = \phi \circ \rho(a)$ for all $a \in A$, that is $\rho'(a) = \phi \circ \rho(a) \circ \phi^{-1}$. Thus the linear map $\rho'(a) : V_2 \to V_2$ is represented by a matrix which is conjugate to the matrix for $\rho(a)$; and this holds for all a (i.e. simultaneous conjugation).

3.2 The ring of generic matrices

In this section we will let $S = k < x_1, x_2, \ldots, x_m >$. Let V be an n-dimensional space, that is $V \cong k^n$. Let $\phi : S \to End(V)$ be a representation of S. As we saw earlier such a representation is decided by giving m $n \times n$ matrices with entries in k since $End(V) \cong M_n(k)$. Let $\Gamma = k[x_{11}^1, \ldots, x_{nn}^m]$ be the commutative polynomial ring in mn^2 variables x_{ij}^l. We then see that the closed points of $Spec(\Gamma)$ correspond to the different n-dimensional representations of S. Let us denote the space of all such representations by χ_S^n. Thus $\chi_S^n \cong \mathbb{A}^{mn^2}$.

Now the group $G = SL_n(k)$ acts on $\bigoplus_m M_n(k)$ by simultaneous conjugation: if $g \in SL_n$ then the action of g on the m-tuple (M_i) is given by

$$g.(M_1, \ldots, M_m) := (gM_1g^{-1}, \ldots, gM_mg^{-1})$$

Since $\bigoplus_m M_n(k)$ is identified (as sets) with χ_S^n, the orbits under this action give us the different isomorphism classes (since isomorphism of the module structures corresponds to the conjugation action).

Consider the map $\Phi : S \to M_n(\Gamma)$ defined by letting $x_l \mapsto (x_{ij}^l)$. The image of this ring homomorphism is called the **ring of generic matrices**.

Now G also acts on Γ. Note that Γ is the ring of regular functions on the variety χ_S^n; that is, the regular functions $\chi_S^n \to k$. The action of a matrix $g \in G$ on such a function γ is given by

$$g.\gamma(\rho) := \gamma(g^{-1}.\rho)$$

for $\rho \in \chi_S^n$. Here the action inside the paranthesis is the action of simultaneous conjugation discussed above. Furthemore, we also have an action on $M_n(\Gamma)$. Again let $g \in G$ and let $M \in M_n(\Gamma)$. Then:

$$g.M(\gamma) := gM(g.\gamma)g^{-1}$$

where γ runs through all the entries of the matrix (and where the action inside the matrix is the one on Γ defined above).

Lemma 3.2.1. *The ring of generic matrices is left invariant under the action of G on $M_n(\Gamma)$.*

Proof. Let $M = \phi(x_k) = (x_{i,j}^k)$ be a generator for the generic matrices, and let $g \in G$. Then g acts on the matrix by:

$$g.(x_{i,j}^k) = gM(g.x_{i,j}^k)g^{-1}$$

where the action inside the matrix is done on each entry separately (as described above). Now if $\rho \in \chi_S^n$ then:

$$g.x_{i,j}^k(\rho) = x_{i,j}^k(g^{-1}.\rho) = x_{i,j}^k(g^{-1}\rho g) = ((g^{-1}\rho g)(x_k))_{ij}$$

so the entry in position ij of $g.M$ is $\gamma_{ij} \in \Gamma$ such that $\gamma_{ij}(\rho) = \rho(x_k)$, hence it is equal to $x_{i,j}^k$ and thus $g.M = M$. It follows from this that any matrix M in $\Phi(S)$ is invariant under this action. □

Now consider again the action of G on Γ. An interesting object to study is then the ring of invariants under this action, namely:

$$\Gamma^G = \Gamma^{SL_n} := \{f \in \Gamma \mid P.f = f \ \forall P \in Sl_n\}$$

Definition 3.2.2. *The trace ring $C_{m,n}$ of m $n \times n$-matrices is defined as the subring of Γ generated by the coefficients of the characteristic polynomials of all matrices in the ring of m generic $n \times n$-matrices.*

It was a conjecture by Artin, later proved by Procesi [6], that $\Gamma^G = C_{m,n}$. In the same paper Procesi also showed that the closed points of Spec$(C_{m,n})$ are in one-to-one correspondence with the equivalence classes of semisimple (n-dimensional) representations of S. However, if we are interested in classifying all equivalence classes there is no general way to describe them as an algebraic scheme.

What if we consider a more general algebra? Let $A = S/I$ where I is an ideal $I = (f_1, \dots, f_r)$. Then the relations in I give rise to a subvariety $\chi_A^n \subset \chi_S^n$, a space parametrizing the different n-dimensional representations of A. If $\Gamma_A :=$ $\Gamma/(\tilde{f}_1, \dots, \tilde{f}_r)$, where $\tilde{f}_j = f_j((x_{p,q}^1), \dots, (x_{p,q}^m))$. then χ_A^n is the variety given by the closed points of Spec(Γ_A). Now one may hope that, as in the case of the free algebra S, the ring of invariants Γ_A^G equals the trace ring of A, and moreover that the trace ring parametrizes the semisimple n-dimensional representations of A (up to isomorphism). But in the general case this need no longer be true!

3.3 Computing the trace ring

If we let $A = S/I$, then the n-trace ring C_n of A is $C_n = C_{m,n}/tr(I)$, where $C(m,n)$ is the trace ring of S and $tr(I)$ is the ideal generated by all expressions of the form $tr(f)$, $f \in I$.

Before we compute some explicit examples, let us state some general facts about the relationship between traces and determinants of 2x2-matrices:

Lemma 3.3.1. *Let $X, Y \in M_2(k)$. Then:*

$$det(X+Y) - det(X) - det(Y) = tr(X) \cdot tr(Y) - tr(XY)$$

Proof. The proof is straightforward: just write up two general matrices and write out the two sides in the equation above, checking that they are equal. □

Lemma 3.3.2. *Let $X, Y \in M_2(k)$. Let t_x, t_y denote the traces of X and Y (respectively), and let d_x, d_y denote the determinants etc. Then:*

1. $X^2 = t_x X - d_x I$
2. $X^3 = (t_x^2 - d_x)X - t_x d_x I$
3. $YX = -XY + t_x Y + t_y X + t_{xy} - t_x t_y$

Proof. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The characteristic polynomial of X is $p(t) = det(X - tI) = (a-t)(d-t) - bc = t^2 - (a+d)t + ad - bc$. By the Cayley-Hamilton theorem, the matrix X satisfies its characteristic equation i.e. we get:

$$X^2 - (a+d)X + (ad-bc)I = O$$

But $a + d = t_x$ and $ad - bc = d_x$, proving the first part. For the second, note that $X^3 = XX^2 = X(t_x X - d_x I) = t_x X^2 - d_x X = t_x(t_x X - d_x I) - d_x X$ and thus this part is also ok.

The last part is proved by first noting that

$$(X+Y)^2 = X^2 + XY + YX + Y^2$$

and then inserting for $X^2, Y^2, (X+Y)^2$ using the first part, plus also using lemma 3.3.1 to eliminate d_{x+y}. The rest is bookkeeping. □

Example 3.3.3. *Suppose $A = k[x] = k < x >$. Then the 2-trace ring C_2 is generated as a k-algebra by t and d, where $t = tr(X)$ and $d = det(X)$ ($X = \phi(x)$, where $\phi : A \to M_2(k)$ is the given representation of A). Hence $C_2 \cong k[t,d]$, the polynomial ring in two (commuting) variables. This follows from lemma 3.3.2 and lemma 3.3.1.*

Example 3.3.4. $A = k[x]/(x^2-1)$. *As in the previous example, let t and d denote the trace and determinant (respectively) of the generic matrix corresponding to x. We have to calculate the ideal $tr(x^2-1)$ of $k[t,d]$. Since the trace of a sum of two matrices equals the sum of the traces, we need only check the traces of x^2-1, $x(x^2-1)$, $x^2(x^2-1)$ etc. We compute:*

$$\begin{aligned} tr(X^2 - I) &= tr(tX - dI - I) = t^2 - 2d - 2 \\ tr(X^3 - X) &= tr((t^2-d)X - tdI - X) = t^3 - td - 2td - t = t^3 - t - 3td \\ tr(X^4 - X^2) &= tr(X^4 - I) = tr((tX - dI)^2 - I) = tr(t^2X^2 - 2tdX + d^2I - I) \\ &= tr((t^3 - 2td)X + (d^2 - t^2d - 1)I) = t^4 - 4t^2d + 2d^2 - 2 \end{aligned}$$

Similarly we compute $tr(X^5 - X^3) = tr(X^5 - X) = t^5 - 5t^3d + 5td^2 - t$, but this last relation (and all subsequent ones) turns out to be superfluous. We can also simplify our relations a little:

$$\begin{aligned} t^3 - t - 3td &= t(t^2 - 3d - 1) = t(t^2 - 2d - 2 - d + 1) = t(0 + 1 - d) = t - td \\ t^4 - 4t^2d + 2d^2 - 2 &= t^2(t^2 - 4d) + 2(d^2 - 1) = t^2(2d + 2 - 4d) + 2(d^2 - 1) \\ &= 2t^2 - 2dt^2 + 2(d^2 - 1) = 2(d^2 - 1) \end{aligned}$$

Hence we find that:

$$C_2 \cong k[t,d]/(t^2 - 2d - 2, td - t, d^2 - 1)$$

The closed points of Spec(C_2) are $(t,d) = (\pm 2, 1)$ and $(t,d) = (0,-1)$. $(t,d) = (\pm 2, 1)$ correspond to:

$$x \mapsto \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

whereas the point $(t,d) = (0,-1)$ correspond to the following two representations:

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, x \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Here the same point corresponds to two different representations, undistinguishable by the trace ring. Note that these module structures are not semi-simple.

Proposition 3.3.5. *Let $A = k < x, y >$. Then the (2-)trace ring is given by $T = T_{2,2} = k[t_x, t_y, d_x, d_y, t_{xy}]$ (where t_x is the trace of the generic matrix associated to x etc.), i.e. a commutative polynomial ring in 5 variables.*

Example 3.3.6. *Let $A = k < x, y > / < x^2 - 1, xy + yx >$. We wish to find the 2-trace ring of A. As above, we have the following relations:*

- $t_x^2 - 2d_x - 2 = 0$
- $t_x - t_x d_x = 0$
- $d_x^2 - 1 = 0$

Here t_x and d_x denotes the trace and determinant (respectively) of the generic matrix corresponding to x. But we also get more:

- $tr(xy + yx) = 2t_{xy} = 0$
- $tr(x(xy+yx)) = tr(2x^2y) = tr(2t_xXY - 2d_xY) = 2t_xt_{xy} - 2d_xt_y = -2d_xt_y = 0$
- $tr(y(xy + yx)) = tr(2xy^2) = -2d_yt_x = 0$

These are all the relations, and by proposition 3.3.5 the 2-trace ring is then:

$$k[t_x, t_y, d_x, d_y, t_{xy}]/(t_x^2 - 2d_x - 2, t_x - t_xd_x, d_x^2 - 1, t_{xy}, d_xt_y, d_yt_x)$$

3.4 Representations of $Ph(A)$

Example 3.4.1. *Let $A = k[x]/(x^2-1)$. Then $Ph(A) = k < x, y > / < x^2-1, xdx + dxx >$ is the ring in the previous example(with dx instead of y). We are interested in finding the 2-dimensional representations of $Ph(A)$. Let $\rho : Ph(A) \to M_2(k)$ be a representation. Up to isomorphism we may assume that $\rho(x)$ is upper triangular, say*

$$X = \rho(x) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

Since $X^2 = I$ we see that $a = \pm 1, c = \pm 1$ and this in turn implies $b = 0$. Hence we have the following possibilities for X:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let $Y = \rho(dx) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Since $XY + YX = 0$ we see that in the first of the four cases above we need to have $XY + YX = IY + YI = 2Y = 0$, i.e. $Y = 0$. Similarly for the second case. In the third case we get:

$$\begin{aligned} XY + YX &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} p & q \\ -r & -s \end{pmatrix} + \begin{pmatrix} p & -q \\ r & -s \end{pmatrix} = \begin{pmatrix} 2p & 0 \\ 0 & -2s \end{pmatrix} = 0 \end{aligned}$$

Hence $Y = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ *for* $q, r \in k$. *The last case is similar (and gives us the same representation, up to isomorphism). Let us look at how this compares with the trace ring found in example 3.3.6 i.e. what are the closed points of the trace ring? We see that one possible choice is* $t_x = 2, d_x = 1, t_y = d_y = 0$ *and this corresponds to the first representation. Similarly, if we look at the point where* $t_x = -2$ *we get the second representation. Both of these representations are simple. The last possibility is if* $t_x = 0$, *and then we will have* $d_x = -1$, $t_y = 0$ *whereas* d_y *can be chosen arbitrarily.*

What is the geometric interpretation of such representations of $Ph(A)$? (A commutative). A 1-dimensional representation is a map $\overline{\rho} : Ph(A) \to End(k) \cong k$, and so we see that it gives us a point $\rho : A \to k$ and a 'tangent vector' at that point (given by what values we assign to the dx_i).

Example 3.4.2. *Consider a map* $\phi : A \to k[\epsilon]/(\epsilon^2)$ *from* A *into the ring of dual numbers. As we have seen, this corresponds to a point and a tangent vector. Now we claim that we have the following* k*-algebra isomorphism:*

$$k[\epsilon]/(\epsilon^2) \cong \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in k\} \subset End(k^2)$$

The isomorphism is given by $a + b\epsilon \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ *(and it is straightforward to show that it is an isomorphism). Hence another way of picking out a point and a tangent vector at that point is to give a 2-dimensional representation* $\rho : A \to End(k^2)$ *where the image* $\rho(a)$ *of each* a *is a matrix of the above form. If we then consider representations* $\overline{\rho} : Ph(A) \to End(k^2)$ *whose restriction to* A *equal* ρ, *we say that such a representation is an infinitesimal tangent over the point in question.*

Example 3.4.3. *If* $\rho : A = k[x]/(x^2 - 1) \to End(k^2)$ *is to be as above, then we have already seen that we will have either* $x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *(corresponding resp to the points* 1 *and* -1*). There are no tangents in this case i.e. if we look at* $\overline{\rho} : Ph(A) \to End(k^2)$ *we require* $dx \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Suppose now that we have two distinct points, which we without loss of generality may take to be 1 and -1 as in example 3.4.1 and example 3.4.3. Based on the above, the following definition makes sense:

Definition 3.4.4. *Let* $A = k[x]/(x^2 - 1)$. *An infinitesimal tangent over the pair of points* $(1, -1)$ *is a* 2*-dimensional representation* $\overline{\rho} : Ph(A) \to End(k^2)$ *that when restricted to* $\rho : A \to End(k^2)$ *takes* x *to* $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *(or more precisely: the equivalence class of such a representation under the conjugation action).*

In our example we see that there are such (nonzero) 'tangents' over our pair of points. This is because of the noncommutativity of the phase space i.e. because $d(x^2 - 1) = xdx + dxx$. If we instead replace $Ph(A)$ by the commutativized $Ph(A)_{com}$ this relation equals $2xdx$ and in that case we see that we get no infinitesimal tangents.

Of course, we might regard our two points as embedded in a larger space. Then we would have a surjective morphism $B \to A$ for some algebra B, and hence we would get an induced surjective morphism $Ph(B) \to Ph(A)$. We could then define an infinitesimal tangent over the pair of points in this space as a representation of $Ph(B)$ that fits together with a representation of $Ph(A)$ as above.

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